

NONCOLLISION SINGULARITIES IN A PLANAR FOUR-BODY PROBLEM

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ABSTRACT. In this paper, we show that there is a Cantor set of initial conditions in a planar four-body problem such that all the four bodies escape to infinity in finite time avoiding collisions. This proves the Painlevé conjecture for the four-body case, thus settles the conjecture completely.

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1. INTRODUCTION

Consider two large bodies Q_1 and Q_2 of masses $m_1 = m_2 = 1$ located at distance χ from each other initially and two small particles Q_3 and Q_4 of masses $m_3 = m_4 = \mu \ll 1$. Q_i s interact with each other via Newtonian potential. We denote the momenta of Q_i by P_i . The Hamiltonian of this system can be written as

$$(1.1) \quad H(Q_1, P_1; Q_2, P_2; Q_3, P_3; Q_4, P_4) = \frac{P_1^2}{2} + \frac{P_2^2}{2} + \frac{P_3^2}{2\mu} + \frac{P_4^2}{2\mu} \\ - \frac{1}{|Q_1 - Q_2|} - \frac{\mu}{|Q_1 - Q_3|} - \frac{\mu}{|Q_1 - Q_4|} - \frac{\mu}{|Q_2 - Q_3|} - \frac{\mu}{|Q_2 - Q_4|} - \frac{\mu^2}{|Q_3 - Q_4|}.$$

We choose the mass center as the origin.

We want to study singular solutions of this system, that are solutions which can not be continued for all positive times. We will exhibit a rich variety of singular solutions. Fix a small ε_0 . Let $\omega = \{\omega_j\}_{j=1}^\infty$ be a sequence of 3s and 4s.

Definition 1.1. *We say that $(Q_i(t), \dot{Q}_i(t))$, $i = 1, 2, 3, 4$, is a **singular solution with symbolic sequence ω** if there exists a positive increasing sequence $\{t_j\}_{j=0}^\infty$ such that*

- $t^* = \lim_{j \rightarrow \infty} t_j < \infty$.
- $|Q_3 - Q_2|(t_j) \leq \varepsilon_0$, $|Q_4 - Q_2|(t_j) \leq \varepsilon_0$.
- If $\omega_j = 4$ then for $t \in [t_{j-1}, t_j]$, $|Q_3 - Q_2|(t) \leq \varepsilon_0$ and $\{Q_4(t)\}_{t \in [t_{j-1}, t_j]}$ winds around Q_1 exactly once.
If $\omega_j = 3$ then for $t \in [t_{j-1}, t_j]$, $|Q_4 - Q_2|(t) \leq \varepsilon_0$ and $\{Q_3(t)\}_{t \in [t_{j-1}, t_j]}$ winds around Q_1 exactly once.
- $|\dot{Q}_i(t)|, \sup_t |Q_i(t)| \rightarrow \infty$ and as $t \rightarrow t^*$, $i = 1, 2, 3, 4$.

During the time interval $[t_{j-1}, t_j]$ we refer to Q_{ω_j} as the traveling particle and to $Q_{7-\omega_j}$ as the captured particle. Thus ω_j prescribes which particle is the traveler during the j trip.

We denote by Σ_ω the set of initial conditions of singular orbits with symbolic sequence ω .

Theorem 1. *There exists $\mu_* \ll 1$ such that for $\mu < \mu_*$ the set $\Sigma_\omega \neq \emptyset$.*

Moreover there is an open set U on the zero energy level and a foliation of U by two-dimensional surfaces such that for any leaf S of our foliation $\Sigma_\omega \cap S$ is a Cantor set.

We remark that the choice of the zero energy level is only for simplicity. Our construction holds for sufficiently small nonzero energy levels.

In [DX] we considered the restricted problem where Q_1 and Q_2 were fixed. So only the sequences where both 3 and 4 appear infinitely many times corresponded to *noncollision* singularities. For sequences where either 3 or 4 appear only finitely many times we had a collision of that body with Q_2 . In the present setting both

Q_1 and Q_2 escape to infinity in finite time so all sequences give noncollision singularities. So in order to simplify notation we assume below that ω is a sequence of all 4s. The general case requires minor modifications as explained in [DX].

1.1. Motivations and perspectives. Our work is motivated by the following fundamental problem in celestial mechanics. *Describe the set of initial conditions of the Newtonian N -body problem leading to global solutions.* The complement to this set splits into the initial conditions leading to the collision and non-collision singularities.

It is clear that the set of initial conditions leading to collisions is non-empty for all $N > 1$ and it is shown in [Sa1] that it has zero measure. Much less is known about the non-collision singularities. The main motivation for our work is provided by following basic problems.

Conjecture 1. *The set of non-collision singularities has zero measure for all $N > 3$.*

This conjecture can be found in [K, Sa3] and in the problem list [Sim] as the first problem. This conjecture remains almost completely open. The only known result is that the conjecture is true if $N = 4$ by Saari [Sa2]. To obtain the complete solution of this conjecture one needs to understand better of the structure of the non-collision singularities. Our Cantor set in Theorem 1 has zero measures and codimension 2 on the energy level (see Remark 2.2), which is in favor of Conjecture 1. As a first step, it is natural to conjecture the following.

Conjecture 2 (Painlevé Conjecture, 1897). *The set of non-collision singularities is non-empty for all $N > 3$.*

There is a long history studying Conjecture 2. There are some nice surveys, see for instance [C, G3, SX] etc. Conjecture 2 probably goes back to Poincaré who was motivated by Sweden King Oscar II prize problem about analytic representation of collisionless solutions of the N -body problem. It was explicitly mentioned in Painlevé's lectures [Pa] where the author proved that for $N = 3$ there are no non-collision singularities using an argument based on triangle inequality (see also [G3] for the argument). Soon after Painlevé, von Zeipel showed that if the system of N bodies has a non-collision singularity then some particle should fly off to infinity in finite time. Thus non-collision singularities seem quite counterintuitive. The first landmark towards proving the conjecture came in 1975. In [MM] Mather and McGehee constructed a system of four bodies on the line where the particles go to infinity in finite time after an infinite number of binary collisions (it was known since the work of Sundman [Su] that binary collisions can be regularized so that the solutions can be extended beyond the collisions). Since Mather-McGehee example had collisions it did not solve Conjecture 2 but made it plausible. Conjecture 2 was proved independently by Xia [X] for the spacial five-body problem and by Gerver [G1] for a planar $3N$ body problem where N is sufficiently large. It is a general belief that a non collision singularity in $(N + 1)$ -body problem can be obtained by adding one more remote and light body to a N -body problem to which the existence of non collision singularities is known. The hardest case of problem, $N = 4$, still remained open. Our result proves the conjecture in the $N = 4$ case.

We believe the method used in this paper could also be used to construct noncollision singularities for general N -body problem, for any $N > 3$. We can put any number of bodies into our system sufficiently far from the mass center of our four bodies orthogonal to the line passing through Q_1 and Q_2 . This produces noncollision singularities in N -body problem. We discuss briefly our approach to arbitrary N case in Section 11. We have not checked all the details in that case but we do not expect any significant difficulties. Treating the general N however would significantly increase the length of the paper, so to simplify the exposition we concentrate here on the four-body case.

Since our technique is perturbative and it is necessary that $\mu \ll 1$, we ask the following questions.

Question 1: *Are there noncollision singularities for a four-body problem in which all the four bodies have comparable masses?*

In fact it is possible that the following stronger result holds.

Question 2: *Is it true that for any choice of positive masses $(m_1, m_2, m_3, m_4) \in \mathbb{R}^3$ the corresponding four-body problem has noncollision singularities?*

We need to develop some nonperturbative techniques for the first question and we need to explore the obstructions for the existence of noncollision singularities for the second.

1.2. Sketch of the proof. The arguments in this paper are similar to the arguments in [DX]. The proof consists of the following three aspects: physical, mathematical and algorithmic aspects. The physical aspect is an idealistic model constructed by [G2] (see Section 2.2), in which the hyperbolic Kepler motion of one light body can extract energy from the elliptic Kepler motion of the other light body. Moreover, after each cycle of energy extraction, the configuration is made self-similar to the beginning, so that the procedure of energy extraction can be iterated infinitely.

The mathematical aspect is a partially hyperbolic dynamics framework that we developed in [DX]. We find that there are two strongly expanding directions that are invariant under iterates along our singular orbits. The strong expansions allow us to push the iteration to the future and synchronize the two light bodies. Namely, the two light bodies can be chosen to come to the correct place simultaneously in order to have close encounter. One of the strong expansion is given by close encounter between Q_1, Q_2 . This is the hyperbolicity created by scattering (hyperbolic Kepler motion). The other one is induced from shears coming from elliptic Kepler motion, which seems quite new in celestial mechanism. See Section 3 and Remark 3.2.

The algorithmic aspect is a systematic toolbox that we develop to compute the derivative of the Poincaré map in details. This toolbox includes symplectic coordinate systems and partition of the phase space (Section 4 and Appendix A), integration of the variational equations (Section 7) and boundary contributions (Section 8), coordinates change between different pieces of the phase space (Section 9), collision exclusion (Section 6.6) and etc. Moreover, we develop new methods to regularize the double collision using hyperbolic Delaunay coordinates and extract \mathcal{C}^1 information of the near double collision from its singular limit, the elastic collision, using

polar coordinates (Section 10). These new methods are more suitable to our framework than the previously known method such that Levi-Civita regularization, and hopefully has wider applications.

Let us also briefly comment on our result in [DX]. In [DX], we consider a model that we call two-center-two-body problem in which the two large bodies Q_1 and Q_2 are fixed. This model, looks artificial, however captures the main difficulties of the problem. Moreover, the calculation is much simpler. The experience and intuition that we get when working on [DX] effectively guide us to go through the harder calculations of this paper.

The paper is organized as follows. In Section 2, we give the proof of the main Theorem 1. In Section 3 we study the structure of the derivative of the local map and the global map. In Section 4, we perform several symplectic transformations to reduce the Hamiltonian system to a form suitable for doing calculations and estimates. This section is purely algebraic without dynamics. Next, we state our estimates for the derivatives of the factor maps of the global map as Proposition 5.1 in Section 5. The following Sections 6, 7, 8 and 9 are devoted to the proof of the proposition. In Appendix C, we give the proof of our main estimate for the derivative of the global map, Lemma 3.2, based on Proposition 5.1. In Section 10, we establish the structure of the local map stated in Lemma 3.1 and prove the nondegeneracy condition. In Section 11, we discuss briefly how to construct non-collision singularities for N -body problem with $N > 4$. Finally, in Appendix A, we give an introduction to Delaunay variables including the estimates of various partial derivatives which are used in our calculations and in Appendix B, we summarize the result of Gerver in [G2] and fill in more detailed proofs.

We briefly talk about the use of computer here. Besides the computer usage in Lemma 3.4, 3.5 and Appendix B.3 that are done in [DX], we use Mathematica in Section 7 and Appendix C to help us multiply matrices in order to prove Lemma 3.2. Multiplying 10×10 matrices involves 1000 multiplications. We need to do such calculations several times. Since we are only interested in main terms such calculations can, in principle, be performed by hand but we believe that the computer is more reliable. Finally we note that the leading terms in the asymptotics of the global map in this paper are the same as in [DX] so the computer is only used to suppress the subleading terms. See Remark C.1 for more discussion.

We use the following conventions for constants.

Convention for constants:

- We use C, c, \hat{C}, \tilde{C} (without subscript) to denote a constant whose value may be different at different contexts.
- When we use subscript 1, 3, 4, for instance C_1, C_3, C_4 , etc, we mean the constant has fixed value throughout the paper specifically chosen for the first, third or fourth body.

2. PROOF OF THE MAIN THEOREM

2.1. The coordinates. We first introduce the set of coordinates needed to state our lemmas and prove our theorems. This set of coordinates is known as the Jacobi coordinates.

Definition 2.1 (The coordinates). • We define the relative position of Q_1, Q_3, Q_4 to Q_2 as the new variables q_1, q_3, q_4

$$(2.1) \quad q_1 = Q_1 - Q_2, \quad q_3 = Q_3 - Q_2, \quad q_4 = Q_4 - Q_2,$$

and the new momentum p_1, p_3, p_4 which is related to the old momentum P_1, P_3, P_4 as

$$(2.2) \quad P_1 = \mu p_1, \quad P_3 = \mu p_3, \quad P_4 = \mu p_4.$$

• Next, we define the new set of variables $(x_3, v_3; x_1, v_1; x_4, v_4)$ called Jacobi coordinates through

$$(2.3) \quad \begin{cases} v_3 = p_3 + \frac{\mu}{1+\mu}(p_4 + p_1), \\ v_1 = p_1, \\ v_4 = p_4 + \frac{\mu p_1}{1+2\mu}, \end{cases} \quad \begin{cases} x_3 = q_3, \\ x_1 = q_1 - \frac{\mu(q_3 + q_4)}{2\mu + 1}, \\ x_4 = q_4 - \frac{\mu q_3}{1+\mu}. \end{cases}$$

One can easily check that this transformation is symplectic, i.e. the following symplectic form $\bar{\omega}$ is preserved

$$(2.4) \quad \bar{\omega} = \sum_{i=3,1,4} dp_i \wedge dq_i = \sum_{i=3,1,4} dv_i \wedge dx_i.$$

This set of new coordinates $(x_3, v_3; x_1, v_1; x_4, v_4)$ look complicated. Heuristically, the new coordinates have the same physical meanings as $(q_3, p_3; q_1, p_1; q_4, p_4)$, since the transformation between them is a $O(\mu)$ perturbation of Id. We introduce the new set of coordinates to simplify the analysis of the variational equation. We will study coordinate changes systematically in Section 4 and 10.

We then use Appendix A to pass to Delaunay variables $(x_3, v_3) \rightarrow (L_3, \ell_3, G_3, g_3)$ and $(x_4, v_4) \rightarrow (L_4, \ell_4, G_4, g_4)$. The variables L_3, L_4 are related to energies of the (x_3, v_3) and (x_4, v_4) systems respectively. We fix the zero energy level such that so that we can eliminate L_4 from our list of variables. Next we pick a Poincaré section and treat ℓ_4 as the new time as we did in [DX] (see Definition 2.3 below), so that we eliminate ℓ_4 from our set of coordinates. So we get $(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4, g_4) \in \mathbb{R}^7 \times \mathbb{T}^3$. This is the set of coordinates that we use to do calculations. In this section and the next section, we use the energy E_3 instead of L_3 , eccentricities e_3, e_4 instead of the angular momentums G_3, G_4 . The new choice of coordinates are related to the old ones through $E_i = \pm \frac{1}{2L_i^2}$, $e_i = \sqrt{1 + 2G_i^2 E_i}$, $i = 3, 4$, + for $i = 4$ and - for $i = 3$. We use the set of coordinates $(E_3, \ell_3, e_3, g_3; x_1, v_1; e_4, g_4)$ to give the proof of the main theorem since it is easier to study their behavior under the renormalization. Actually, our system still has total angular momentum conservation. We could have fixed an angular momentum and eliminated two more variables. However, this would lead to more complicated formulas.

Notation 2.2. • We refer to our set of variables as $\mathcal{V} = (\mathcal{V}_3; \mathcal{V}_1; \mathcal{V}_4) = (L_3, \ell_3, G_3, g_3; x_1, v_1; G_4, g_4)$.

- We denote the Cartesian variables as $\mathcal{X} := (\mathcal{X}_3; \mathcal{X}_1; \mathcal{X}_4) = (x_3, v_3; x_1, v_1; x_4, v_4)$.
- In the following, when we use Cartesian coordinates such as x, v , each letter has two components. We will use the subscript \parallel to denote the horizontal coordinate and subscript \perp to denote the vertical coordinate. So we write $x = (x_{\parallel}, x_{\perp})$ and $v = (v_{\parallel}, v_{\perp})$, etc.

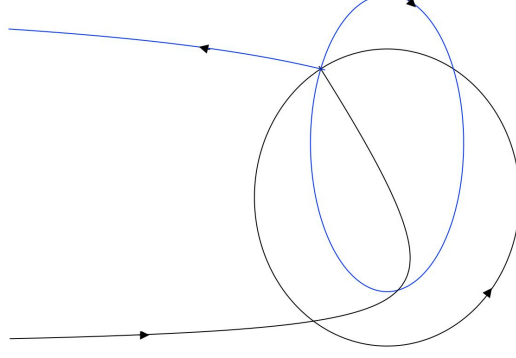


FIGURE 1. Angular momentum transfer

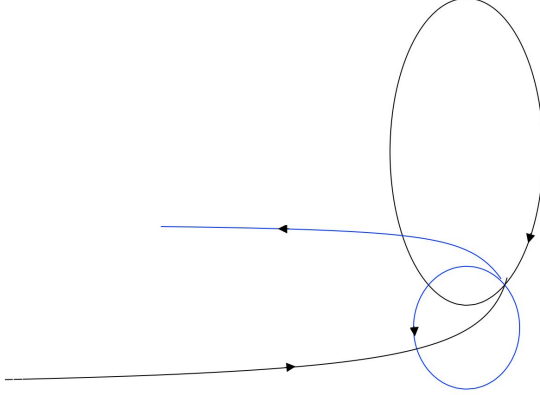


FIGURE 2. Energy transfer

2.2. Gerver's model. Following [G2], we discuss in this section the limit case $\mu = 0, \chi = \infty$. We assume that Q_3 has elliptic motion and Q_4 has hyperbolic motion with focus Q_2 . Since $\mu = 0$, Q_3 and Q_4 do not interact unless they have exact collision. We also assume the traveler always has horizontal asymptotes, i.e. the slopes of incoming asymptote θ^- and that of the outgoing asymptote $\bar{\theta}^+$ of the traveler particle should satisfy $\theta^- = 0$, $\bar{\theta}^+ = \pi$.

The Gerver map describes the parameters of the elliptic orbit change during the interaction of Q_3 and Q_4 . The orbits of Q_3 and Q_4 intersect in two points. We pick one of them. We use subscript $j \in \{1, 2\}$ to describe the first or the second collision in Gerver's construction.

Since Q_3 and Q_4 only interact when they are at the same point and energy and momentum are conserved, the interaction is described by the elastic collision. That is, velocities before and after the collision are related by

$$(2.5) \quad v_3^+ = \frac{v_3^- + v_4^-}{2} + \left| \frac{v_3^- - v_4^-}{2} \right| n(\alpha), \quad v_4^+ = \frac{v_3^- + v_4^-}{2} - \left| \frac{v_3^- - v_4^-}{2} \right| n(\alpha),$$

where $n(\alpha)$ is a unit vector making angle α with $v_3^- - v_4^-$.

With this in mind we proceed to define the Gerver map $\mathbf{G}_{e_4, j, \omega}(E_3, e_3, g_3)$. This map depends on two discrete parameters $j \in \{1, 2\}$ and $\omega \in \{3, 4\}$. The role of j has been explained above, and ω will tell us which particle will be the traveler after the collision.

To define \mathbf{G} we assume that Q_4 moves along the hyperbolic orbit with parameters $(-E_3, e_4, g_4)$ where g_4 is fixed by requiring that the incoming asymptote of Q_4 is horizontal. We assume that Q_3 and Q_4 arrive to the j -th intersection point of their orbit simultaneously. At this point their velocities are changed by (2.5) where the only free parameter α is fixed by the condition $\bar{\theta}_\omega = \pi$.

After that the particle proceed to move independently. Thus Q_3 moves on an orbit with parameters $(\bar{E}_3, \bar{e}_3, \bar{g}_3)$, and Q_4 moves on an orbit with parameters $(\bar{E}_4, \bar{e}_4, \bar{g}_4)$.

If $\omega = 4$, we choose α so that after the exchange Q_4 moves on hyperbolic orbit and $\bar{\theta}_4^+ = \pi$ and let

$$\mathbf{G}_{e_4, j, 4}(E_3, e_3, g_3) = (\bar{E}_3, \bar{e}_3, \bar{g}_3).$$

If $\omega = 3$ we choose α so that after the exchange Q_3 moves on hyperbolic orbit and $\bar{\theta}_3^+ = \pi$ and let

$$\mathbf{G}_{e_4, j, 3}(E_3, e_3, g_3) = (\bar{E}_4, \bar{e}_4, \bar{g}_4).$$

In the following, to fix our notation, we always call the captured particle Q_3 and the traveler Q_4 .

We will denote the ideal orbit parameters in Gerver's paper [G2] of Q_3 and Q_4 before the first (respectively second) collision with $*$ (respectively $**$). Thus, for example, G_4^{**} will denote the angular momentum of Q_4 before the second collision. The real values after the first (respectively, after the second) collisions are denoted with a bar or double bar.

Gerver's map \mathbf{G} has a skew product form

$$\bar{e}_3 = f_e(e_3, g_3, e_4), \quad \bar{g}_3 = f_g(e_3, g_3, e_4), \quad \bar{E}_3 = E_3 f_E(e_3, g_3, e_4).$$

This skew product structure will be crucial in the proof of Theorem 1 since it will allow us to iterate \mathbf{G} so that E_3 grows exponentially while e_3 and g_3 remain almost unchanged.

The following fact plays a key role in constructing singular solutions.

Lemma 2.1 ([G2], see Lemma 2.1 of [DX]). *There exist (e_3^*, g_3^*) , $\sqrt{2}/2 < e_3^* < 1$, such that for sufficiently small $\bar{\delta} > 0$ given $\omega', \omega'' \in \{3, 4\}$, there exist $\lambda_0 > 1$ and functions $e_4'(e_3, g_3)$, $e_4''(e_3, g_3)$, defined in a small (depending on $\bar{\delta}$) neighborhood of (e_3^*, g_3^*) , such that*

$$(a) \text{ for } e_4^*, e_4^{**} \text{ given by } e_4'(e_3^*, g_3^*) = e_4^* \text{ and } e_4''(e_3^*, g_3^*) = e_4^{**}, \text{ we have}$$

$$(e_3, g_3, E_3)^{**} = \mathbf{G}_{e_4^*, 1, \omega'}(e_3, g_3, E_3)^*, \quad (e_3, -g_3, \lambda_0 E_3)^* = \mathbf{G}_{e_4^{**}, 2, \omega''}(e_3, g_3, E_3)^{**},$$

- (b) If (e_3, g_3) lie in a $\bar{\delta}$ neighborhood of (e_3^*, g_3^*) , we have
- $$(\bar{e}_3, \bar{g}_3, \bar{E}_3) = \mathbf{G}_{e_4'(e_3, g_3), 1, \omega'}(e_3, g_3, E_3), \quad (\bar{\bar{e}}_3, -\bar{\bar{g}}_3, \bar{\bar{E}}_3) = \mathbf{G}_{e_4''(e_3, g_3), 2, \omega''}(\bar{e}_3, \bar{g}_3, \bar{E}_3),$$
- and $\bar{e}_3 = e_3^*$, $\bar{g}_3 = g_3^*$, $\bar{E}_3 = \lambda(e_3, g_3)E_3$ where $\lambda_0 - \bar{\delta} < \lambda < \lambda_0 + \bar{\delta}$.

Part (a) is the main content of [G2], which gives a two-step procedure to decrease the energy of the elliptic Kepler motion and maintain the self-similar structure (See Figure 1 and 2). The results are summarized in Appendix B. Part (b) says that once the ellipse gets deformed slightly away from the standard case in Figure 1 after the first collision (angler momentum transfer), we can correct it by changing the phase of Q_3 slightly at the next collision to guarantee the ellipse that we get after the second collision (energy transfer) is standard. This idea was proposed in [G2]. We fill in details of the proof in Appendix B.3.

2.3. The local and global map, and the renormalization map.

Definition 2.3 (The Poincaré section, the local map, the global map and the Poincaré map). *We define a section $\{x_{4,\parallel} = -2\}$. To the right of this section, we define the local map*

$$\mathbb{L} : \{x_{4,\parallel} = -2, v_{4,\parallel} > 0\} \rightarrow \{x_{4,\parallel} = -2, v_{4,\parallel} < 0\},$$

and to the left global map

$$\mathbb{G} : \{x_{4,\parallel} = -2, v_{4,\parallel} < 0\} \rightarrow \{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}.$$

Finally, we define the Poincaré map

$$\mathcal{P} = \mathbb{G} \circ \mathbb{L} : \{x_{4,\parallel} = -2, v_{4,\parallel} > 0\} \rightarrow \{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}.$$

These maps $\mathbb{G}, \mathbb{L}, \mathcal{P}$ are defined by the standard procedure following the Hamiltonian flow. Once we find one orbit going from one section to another, the corresponding map can be defined in a neighborhood of this orbit. The existence of returning orbit is contained in the following Lemma 2.7.

Next, we define the renormalization map \mathcal{R} . The renormalization map \mathcal{R} manifests itself in Lemma 2.4 and its proof.

Definition 2.4 (The renormalization map). *We define the renormalization map \mathcal{R} in several steps as follows. This definition depends on a parameter χ which can be thought as a typical distance between the heavy bodies Q_1 and Q_2 .*

- *Partition: we partition of section $\{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}$ into cubes of size $1/\sqrt{\chi}$ for $\chi \gg 1$ being the initial value of $|x_{1,\parallel}|$.*
- *Dilation: we zoom in the configuration space by $\lambda > 1$, where $1/\lambda$ is the semimajor axis of the ellipse measured for the center point in each $1/\sqrt{\chi}$ cube.*
- *Rotation: we rotate the x -axis around Q_2 , so that for the center point in each $1/\sqrt{\chi}$ cube, we have that $x_{1,\perp} = 0$. We will prove in Lemma 2.4 and 6.2 that the angle of rotation is $O(\mu/\chi^{1/2})$. We denote by $\text{Rot}(\beta)$ the rotation of the plane by angle β .*
- *Reflection: we reflect the whole system along the x -axis.*
- *Iteration: finally we set χ to be equal to the distance between Q_1 and Q_2 for the orbit in the center of each cube.*

\mathcal{R} will be applied after two applications of the Poincaré map.

We push forward each cube to the section $\{x_{4,\parallel} = -2/\lambda, v_{4,\parallel} > 0\}$. We include the piece of orbits from the section $\{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}$ to $\{x_{4,\parallel} = -2/\lambda, v_{4,\parallel} > 0\}$ to the global map \mathbb{G} and apply the \mathcal{R} to the section $\{x_{4,\parallel} = -2/\lambda, v_{4,\parallel} > 0\}$. So the locally constant map \mathcal{R} amounts to zooming in the configuration by multiplying it by λ , slowing down the velocity by dividing it by $\sqrt{\lambda}$, and then applying a rotation and a reflection. We have

$$(2.6) \quad \begin{aligned} \mathcal{R}(\{x_{4,\parallel} = -2/\lambda, v_{4,\parallel} > 0\}) &= \{(\text{Rot}(\beta)^{-1} \cdot x_4)_{\parallel} = -2, v_{4,\parallel} > 0\}, \quad \text{and} \\ \mathcal{R}(E_3, \ell_3, e_3, g_3; x_1, v_1; e_4, g_4) &= \left(\frac{E_3}{\lambda}, \ell_3, e_3, -(g_3 - \beta); \right. \\ &\quad \left. \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{Rot}(\beta)x_1, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\text{Rot}(\beta)v_1}{\sqrt{\lambda}}; e_4, -(g_4 - \beta) \right). \end{aligned}$$

We will see in Lemma 2.4 that $\beta = O(\mu/\sqrt{\chi})$ hence the section $\{(\text{Rot}(\beta)^{-1} \cdot x_4)_{\parallel} = -2, v_{4,\parallel} > 0\}$ forms a small $O(\mu/\chi^{1/2})$ angle with the section $\{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}$ when seeing in the configuration space. This difference is negligible, so we always talk about the section $\{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}$ for notational simplicity.

The renormalization ensures that the semimajor axis of the Q_3 ellipse is $1 + O(\mu)$. We shall show that for orbits of interest \mathcal{R} sends χ to $\lambda\chi(1 + O(\mu))$. Thus χ will grow to infinity exponentially. Therefore without loss of generality we always assume in our estimates that $1/\chi \ll \mu$.

2.4. Asymptotics of the local and global map. In the next two lemmas, our notations are such that \mathbb{L}, \mathbb{G} send *unbarred* variables to *barred* variables. We use θ to denote the slope of the asymptote of x_4, v_4 and the superscripts $+$ (or $-$) to specify the orbit parameters entering (or exiting) the close encounter between Q_3 and Q_4 .

The case of zero total angular momentum differs from the nonzero case drastically.

Definition 2.5. We introduce the following quantity \mathcal{G}_χ to deal with the both cases simultaneously

$$(2.7) \quad \mathcal{G}_\chi = \begin{cases} 1 & \text{if the total angular momentum is 0,} \\ \bar{\varepsilon}\sqrt{\chi} & \text{if the total angular momentum is } G_0 \neq 0, \end{cases}$$

where $\bar{\varepsilon}$ is a small constant. We omit the subscript χ , when there is no danger of confusion.

We note that the case of zero angular momentum is much simpler so the readers who only want to see the construction of the non collision singularities in the four body case may concentrate on that case only. The general case is only needed when we explain how to add more bodies to the system.

To simplify the presentation, we list standard assumptions that we will impose on the initial or final values of the local and global map respectively.

AL:

(AL.1) Initially in the section $\{x_{4,\parallel} = -2, v_{4,\parallel} > 0\}$ we have for some $\delta \ll 1$ independent of χ, μ that

$$\left| E_3(0) + \frac{1}{2} \right| \leq C_3 \delta, \quad \sqrt{2}/2 < e_3(0) < 1.$$

(AL.2) The incoming and outgoing asymptotes of the nearly hyperbolic motion of x_4, v_4 satisfy

$$\theta_4^- = O(\mu), \quad \text{and } |\bar{\theta}_4^+ - \pi| \leq \tilde{\theta} \ll 1,$$

where $\tilde{\theta}$ is independent of μ, χ .

(AL.3) The initial value of x_1, v_1 satisfies $|x_1| \geq \chi - \frac{1}{\sqrt{\chi}}, |v_1| \leq C$.

AG:

(AG.1) Initially in the section $\{x_{4,\parallel} = -2, v_{4,\parallel} < 0\}$, we have for some $\delta \ll 1$ independent of χ, μ that

$$|E_3(0) - E_3^{*i}| \leq C_3 \delta, \quad \sqrt{2}/2 < e_3(0) < 1,$$

where E_3^{*i} , $i = 1, 2$ is the energy of E_3 after the first ($i = 1$, $E_3^{*1} = -1/2$) or second collision ($i = 2$) in Gerver's model (see Appendix B and [G2]).

(AG.2) On the section $\{x_{4,\parallel} = -2\}$, we have $|x_{4,\perp}| < C_4$ holds both at initial and final moments.

(AG.3) The initial condition of x_1, v_1 satisfies

$$(2.8) \quad -2\chi \leq x_{1,\parallel} \leq -\chi + \frac{1}{\sqrt{\chi}}, \quad |x_{1,\perp}| \leq C\mu\mathcal{G}, \quad v_1 = O(1, \mathcal{G}/\chi), \quad -\bar{c} \leq v_{1,\parallel} < -c.$$

The next lemma shows that the real local map \mathbb{L} is well approximated by \mathbf{G} in the \mathcal{C}^0 sense. Its proof will be given in Section 10.

Lemma 2.2. *Assume **AL**. Then after the application of \mathbb{L} , the following asymptotics hold uniformly*

$$(\bar{E}_3, \bar{e}_3, \bar{g}_3) = \mathbf{G}_{e_4}(E_3, e_3, g_3) + o(1).$$

as $1/\chi \ll \mu$, $\tilde{\theta} \rightarrow 0$.

The assumptions are met due to the following Lemma 2.3 and 2.4.

The next lemma deals with the \mathcal{C}^0 estimates for the global map \mathbb{G} .

Lemma 2.3. *Assume **AG**. Then after the application of \mathbb{G} the following estimates hold uniformly in χ, μ*

$$\begin{aligned} (a) \quad & \frac{\bar{E}_3}{E_3} - 1 = O(\mu), \quad \frac{\bar{G}_3}{G_3} - 1 = O(\mu), \quad \bar{g}_3 - g_3 = O(\mu), \\ (b) \quad & \theta_4^+ = \pi + O(\mu), \quad \bar{\theta}_4^- = O(\mu). \end{aligned}$$

The proof of this lemma is given in Section 6.

The next lemma deals with the \mathcal{C}^0 estimates of x_1, v_1 . It is convenient to study the map $\mathcal{R} \circ \mathcal{P}^2$ directly. The proof is also in Section 6.

Lemma 2.4. *There exist constants $c_1, \bar{c}_1, C_1, \bar{C}_1, \tilde{C}_1 > 0$ independent of μ, χ , such that the following holds. Given C_3, C_4, δ , consider the orbit satisfying*

(i) $|x_{4,\perp}| < C_4$ for the first four times the orbits visit the section $\{x_{4,\parallel} = -2\}$,

- (ii) for all time $|E_3 - E_3^{*i}| \leq 2C_3\delta$, $\sqrt{2}/2 < e_3 < 1$, for some $\delta \ll 1$ independent of χ, μ , where E_3^{*i} is as the first bullet point of **AG**.
 - (iii) the total angular momentum $|G| \leq \mathcal{G}_\chi$.
 - (iv) on the section $\{x_{4,\parallel} = -2\}$, initially $(x_1, v_1)(0)$ satisfy
- $$(2.9) \quad -\chi - \frac{1}{\sqrt{\chi}} \leq x_{1,\parallel}(0) \leq -\chi + \frac{1}{\sqrt{\chi}}, \quad |x_{1,\perp}(0)| \leq \frac{1}{\sqrt{\chi}}, \quad -\bar{c}_1 \leq v_{1,\parallel}(0) \leq -c_1,$$

then

- (a) after the application of $\mathcal{P}, \mathcal{P}^2$, estimates (2.8) hold for some c, \bar{c}, C .
- (b) after the application of \mathcal{P}^2 (variables carry double bar), we have

$$\tilde{C}_1^{-1}\mu \leq \frac{\bar{\bar{x}}_{1,\parallel}}{x_{1,\parallel}} - 1 \leq \tilde{C}_1\mu, \quad \left| \frac{\bar{\bar{x}}_{1,\perp}}{\bar{\bar{x}}_{1,\parallel}} \right| \leq \frac{\bar{C}_1\mu\mathcal{G}_\chi}{\chi}.$$

- (c) after the application of $\mathcal{R} \circ \mathcal{P}^2$, we get the renormalized χ , denoted by $\tilde{\chi}$, satisfies $\lambda(1 + \tilde{C}_1^{-1}\mu)\chi \leq \tilde{\chi} \leq \lambda(1 + \tilde{C}_1\mu)\chi$, and we have $|\mathcal{R}(\bar{\bar{G}})| \leq \mathcal{G}_{\tilde{\chi}}$ and

$$(2.10) \quad \begin{aligned} -\tilde{\chi} - \frac{1}{\sqrt{\tilde{\chi}}} &\leq \mathcal{R}(\bar{\bar{x}}_{1,\parallel}) \leq -\tilde{\chi} + \frac{1}{\sqrt{\tilde{\chi}}}, \quad |\mathcal{R}(\bar{\bar{x}}_{1,\perp})| \leq \frac{1}{\sqrt{\tilde{\chi}}}, \\ -\bar{c}_1 &\leq \mathcal{R}(\bar{\bar{v}}_{1,\parallel}) \leq -c_1, \quad |\mathcal{R}(\bar{\bar{v}}_{1,\perp})| \leq C_1 \frac{\mathcal{G}_{\tilde{\chi}}}{\tilde{\chi}}, \end{aligned}$$

where λ is the renormalization factor in Definition 2.4.

Remark 2.1. Part (b) also implies that $|\tan \beta| \leq \frac{\bar{C}_1\mu\mathcal{G}_\chi}{\chi}$, where β is the rotation angle in Definition 2.4. As $\chi \rightarrow \infty$ exponentially, this rotation angle decays to zero exponentially.

2.5. Invariant cones and the proof of the main theorem. We define the following two sets.

Definition 2.6. Consider the part of phase space with total angular momentum $|G| < C$ for some constant C . Given δ consider open sets defined by

$$\begin{aligned} U_1(\delta) &= \left\{ (2.9) \text{ holds and } \left| E_3 + \frac{1}{2} \right|, |e_3 - e_3^*|, |g_3 - g_3^*|, |\theta_4^-| < \delta, |e_4 - e_4^*| < \sqrt{\delta} \right\}, \\ U_2(\delta) &= \left\{ (2.9) \text{ holds and } |E_3 - E_3^{**}|, |e_3 - e_3^{**}|, |g_3 - g_3^{**}|, |\theta_4^-| < \delta, |e_4 - e_4^{**}| < \sqrt{\delta} \right\}. \end{aligned}$$

In this definition we do not restrict ℓ_3 since ℓ_3 can achieve any value in $[0, 2\pi]$, as will be seen in Lemma 2.7 below.

The next lemma establishes (partial) hyperbolicity of the Poincaré map.

Lemma 2.5. There are cone families \mathcal{K}_1 on $T_{U_1(\delta)}(\mathbb{R}^7 \times \mathbb{T}^3)$ and \mathcal{K}_2 on $T_{U_2(\delta)}(\mathbb{R}^7 \times \mathbb{T}^3)$, each of which contains a two dimensional plane and a constant c such that for all $\mathbf{x} \in U_1(\delta)$ satisfying $\mathcal{P}(\mathbf{x}) \in U_2(\delta)$, and for all $\mathbf{x} \in U_2(\delta)$ satisfying $\mathcal{R} \circ \mathcal{P}(\mathbf{x}) \in U_1(\delta)$, we have

- (a) $d\mathcal{P}(\mathcal{K}_1) \subset \mathcal{K}_2$, $d(\mathcal{R} \circ \mathcal{P})(\mathcal{K}_2) \subset \mathcal{K}_1$.
- (b) If $v \in \mathcal{K}_1$, then $\|d\mathcal{P}(v)\| \geq c\chi\|v\|$.
If $v \in \mathcal{K}_2$, then $\|d(\mathcal{R} \circ \mathcal{P})(v)\| \geq c\chi\|v\|$.

We give the proof in Section 3.

We call a \mathcal{C}^1 surface $S_1 \subset U_1(\delta)$ (respectively $S_2 \subset U_2(\delta)$) **admissible** if $TS_1 \subset \mathcal{K}_1$ (respectively $TS_2 \subset \mathcal{K}_2$).

- Lemma 2.6.** (a) The vector $\tilde{w} = \frac{\partial}{\partial \ell_3}$ is in \mathcal{K}_i .
 (b) Any plane Π in \mathcal{K}_i the projection map $\pi_{e_4, \ell_3} = (de_4, d\ell_3) : \Pi \rightarrow \mathbb{R}^2$ is one-to-one. In other words (e_4, ℓ_3) can be used as coordinates on admissible surfaces.

We call an admissible surface **essential** if π_{e_4, ℓ_3} is an $I \times \mathbb{T}^1$ for some interval I . In other words given $e_4 \in I$ we can prescribe ℓ_3 arbitrarily.

- Lemma 2.7.** (a) Given an essential admissible surface $S_1 \subset U_1(\delta)$ and $\tilde{e}_4 \in I(S_1)$ there exists $\tilde{\ell}_3$ such that $\mathcal{P}((\tilde{e}_4, \tilde{\ell}_3)) \in U_2(\delta)$. Moreover if $\text{dist}(\tilde{e}_4, \partial I) > 1/\chi$ then there is a neighborhood $V(\tilde{e}_4)$ of $(\tilde{e}_4, \tilde{\ell}_3)$ such that $\pi_{e_4, \ell_3} \circ \mathcal{P}$ maps V surjectively to

$$\{|e_4 - e_4^*| < K\delta\} \times \mathbb{T}^1.$$

- (b) Given an essential admissible surface $S_2 \subset U_2(\delta)$ and $\tilde{e}_4 \in I(S_2)$ there exists $\tilde{\ell}_3$ such that $\mathcal{R} \circ \mathcal{P}((\tilde{e}_4, \tilde{\ell}_3)) \in U_1(\delta)$. Moreover if $\text{dist}(\tilde{e}_4, \partial I) > 1/\chi$ then there is a neighborhood $V(\tilde{e}_4)$ of $(\tilde{e}_4, \tilde{\ell}_3)$ such that $\pi_{e_4, \ell_3} \circ \mathcal{R} \circ \mathcal{P}$ maps V surjectively to

$$\{|e_4 - e_4^{**}| < K\delta\} \times \mathbb{T}^1.$$

- (c) For points in $V(\tilde{e}_4)$ from parts (a) and (b), the particles avoid collisions before the next return and the minimal distance between Q_3 and Q_4 satisfies $\mu\delta \leq d \leq \frac{\mu}{\delta}$.

Proof of the main Theorem 1. We will iterate $\mathcal{R} \circ \mathbb{G} \circ \mathbb{L} \circ \mathbb{G} \circ \mathbb{L}$. The conclusion of Lemma 2.2 implies the (AG.1), i.e. the assumption of 2.3. The (AG.2) is about the existence of returning orbit for the global map, which is given by Lemma 2.7. (AG.3) is given by Lemma 2.4. The conclusions of Lemma 2.3 imply the (AL.1), (AL.2), which are the key assumptions of Lemma 2.2. (AL.3) is also given by Lemma 2.4.

Fix a number ε which is small but is much larger than both μ and $1/\chi$. Let S_0 be an admissible surface such that the diameter of S_0 is much larger than $1/\chi$ and such that on S_0 we have

$$|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.$$

where (\hat{e}_3, \hat{g}_3) is close to (e_3^*, g_3^*) . For example, we can pick a point $\mathbf{x} \in U_1(\delta)$ and let \hat{w} be a vector in $\mathcal{K}_1(\mathbf{x})$ such that $\frac{\partial}{\partial \ell_3}(\hat{w}) = 0$. Then let

$$S_0 = \{(E_3, \ell_3, e_3, g_3; x_1, v_1; e_4, g_4)(\mathbf{x}) + a\hat{w} + (0, b, 0_{1 \times 8})\}_{a \leq \varepsilon/\bar{K}}$$

where \bar{K} is a large constant.

We wish to construct a singular orbit in S_0 . We define S_j inductively so that S_j is component of $\mathcal{P}(S_{j-1}) \cap U_2(\delta)$ if j is odd and S_j is component of $(\mathcal{R} \circ \mathcal{P})(S_{j-1}) \cap U_1(\delta)$ if j is even (we shall show below that such components exist). Let $\mathbf{x} = \lim_{j \rightarrow \infty} (\mathcal{R}\mathcal{P}^2)^{-j} S_{2j}$. We claim that \mathbf{x} has singular orbit. Indeed by Lemma 2.1 the unscaled energy of (x_3, v_3) satisfies $-E_3(j) \geq (\lambda_0 - \tilde{\delta})^{j/2}$ where $\tilde{\delta} \rightarrow 0$ as $\delta \rightarrow 0$.

According to Lemma 2.3, Lemma 2.4 and the total energy conservation, we get the velocity of $|v_4|$ during the trip j is bounded from below by $c\sqrt{-E_3(j)} \geq c(\lambda_0 - \tilde{\delta})^{j/4}$. Note that by definition of $U_1(\delta)$ the initial conditions for x_1, v_1 are chosen to satisfy the assumption (2.9). Lemma 2.4 then shows that the assumptions on x_1, v_1 are always satisfied. Thus we can iterate Lemma 2.4 for arbitrarily many steps. Now let us look at the orbit in the physical space without doing any renormalization. Inductively, we have

$$(x_{1,\parallel})_j \in \left[-(1 + \mu\tilde{C}_1)^j \chi_0, -(1 + \mu\tilde{C}_1^{-1})^j \chi_0 \right]$$

after j -th applications of \mathcal{P}^2 using part (b) of Lemma 2.4. Therefore, $x_{1,\parallel} \rightarrow -\infty$ as $n \rightarrow \infty$. Next

$$t_{j+1} - t_j = O((\lambda_0 - \tilde{\delta})^{-j/4}) |(x_{1,\parallel})_j|$$

so the total time is bounded by the sum

$$t_* = \lim_{j \rightarrow \infty} t_j \leq \text{Const} \chi_0 \sum_{j=1}^{\infty} \frac{1}{(\lambda_0 - \tilde{\delta})^{j/4}} (1 + \mu\tilde{C}_1)^j < \infty$$

as needed. This shows that infinite many steps complete within finite time and x_1 goes to infinity. Since μ is small, from (2.3), we see that q_1 also goes to infinity. This implies that both Q_1 and Q_2 escape to infinity since $q_1 = Q_1 - Q_2$ and the mass center is fixed. We also have that Q_3 escapes to infinity since Q_3 is close to Q_2 , i.e. q_3 is bounded. Finally, Q_4 travels between Q_1 and Q_2 .

It remains to show that if we can find a component of $\mathcal{P}(S_{2j})$ inside $U_2(\delta)$ and a component of $(\mathcal{R} \circ \mathcal{P}(S_{2j+1}))$ inside $U_1(\delta)$. For the sake of completeness, we cite it here. Note that Lemma 2.7 allows to choose such components inside larger sets $U_2(K\delta)$ and $U_1(K\delta)$.

First note that by Lemma 2.3 on $\mathcal{P}(S_{2j}) \cap U_1(K\delta)$ and on $(\mathcal{R} \circ \mathcal{P}^2)(S_{2j}) \cap U_2(K\delta)$ we have $\theta_4^- = O(\mu)$. Also by Lemma 2.7 e_4 can be prescribed arbitrarily. In other words we have a good control on the orbit of Q_4 .

In order to control the orbit of Q_3 note that by Lemma 2.5(b) the preimage of S_{2j} has size $O(1/\chi)$ and so by Lemmas 2.2, 2.3 and 2.6 given ε we have that e_3 and g_3 have oscillation less than ε on S_{2j} if μ is small enough. Namely part (b) of Lemma 2.6 shows that e_3 and g_3 have oscillation $O(1/\chi)$ on the preimage of S_{2j} while Lemmas 2.2 and 2.3 show that the oscillations do not increase much after application of local and global map. Thus there exist (\hat{e}_3, \hat{g}_3) such that on S_{2j} we have

$$|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.$$

Also due to rescaling in the definition of \mathcal{R} and Lemma 2.3, we have

$$\left| E_3 - \left(-\frac{1}{2} \right) \right| = O\left(\frac{1}{\sqrt{\chi}} + \mu \right).$$

Set

$$(2.11) \quad \tilde{S}_{2j+1} = \mathcal{P}V(e'(\hat{e}_3, \hat{g}_3)), \quad \tilde{S}_{2j+2} = (\mathcal{R} \circ \mathcal{P})V(e''(\hat{e}_3, \hat{g}_3)).$$

Then on \tilde{S}_{2j+1} we shall have

$$|e_3 - e_3^{**}| < K\varepsilon, \quad |g_3 - g_3^{**}| < K\varepsilon \text{ and } |E_3 - E_3^{**}| < K\varepsilon$$

while on \tilde{S}_{2j+2} we shall have

$$|e_3 - e_3^*| < K^2\varepsilon, \quad |g_3 - g_3^*| < K^2\varepsilon \text{ and } \left| E_3 + \frac{1}{2} \right| < K(1/\sqrt{\chi} + \mu).$$

Denote

$$S_{2j+1} = \tilde{S}_{2j+1} \cap \{|e_4 - e''(e_3^*, g_3^*)| < \sqrt{\delta}\}, \quad S_{2j+2} = \tilde{S}_{2j+2} \cap \{|e_4 - e'(e_3^*, g_3^*)| < \sqrt{\delta}\}.$$

Taking ε so small that $K^2\varepsilon < \delta$ we get that $S_{2j+1} \in U_2(\delta)$, $S_{2j+2} \in U_1(\delta)$ as needed.

Finally we use the freedom to choose the appropriate partition in the definition of \mathcal{R} to ensure that \mathcal{R} is continuous on the preimage of $V(e'(\hat{e}_3, \hat{g}_3))$ so that $V(e'(\hat{e}_3, \hat{g}_3))$ is a smooth surface. \square

Remark 2.2. *In each step of the Cantor set construction above, the ratio of the remained measure with the deleted measure is $O(1/\chi)$ as $\chi \rightarrow \infty$ and μ, δ fixed. As the expansion rate χ grows exponentially under iterates due to the renormalization, the resulting Cantor set on each piece of admissible surface is a zero Hausdorff dimension set.*

3. THE HYPERBOLICITY OF THE POINCARÉ MAP

In this section, we consider the hyperbolicity of the Poincaré map by studying the derivative of the local and global maps.

3.1. The structure of the derivative of the global map and local map.

Lemma 3.1. *Suppose $\mathbf{x} \in U_j(\delta)$ and $\mathbb{L}(\mathbf{x})$ satisfy $\theta_4^- = O(\mu)$, $|\bar{\theta}_4^+ - \pi| \leq \tilde{\theta} \ll 1$. Then there exist a linear functional \mathbf{l}_j and a vector \mathbf{u}_j such that*

$$d\mathbb{L}(\mathbf{x}) = \frac{1}{\mu} \mathbf{u}_j(\mathbf{x}) \otimes \mathbf{l}_j(\mathbf{x}) + B(\mathbf{x}) + o(1),$$

and we have

$$\mathbf{l}_j = \hat{\mathbf{l}}_j + o(1), \quad \mathbf{u}_j = \hat{\mathbf{u}}_j + o(1), \quad B = \hat{B}_j + o(1), \text{ as } \delta, \tilde{\theta}, 1/\chi \ll \mu \rightarrow 0,$$

where $j = 1, 2$ meaning the first or the second collision.

The proof is given in Section 10.

Lemma 3.2. *Let \mathbf{x} and $\mathbf{y} \in \mathbb{G}(\mathbf{x}) \in U_{3-j}(\delta)$ be the initial and final values of the global map \mathbb{G} and satisfy **AG**. Then there exist linear functionals $\bar{\mathbf{l}}(\mathbf{x})$ and $\bar{\bar{\mathbf{l}}}(\mathbf{x})$ and vectorfields $\bar{\mathbf{u}}(\mathbf{y})$ and $\bar{\bar{\mathbf{u}}}(\mathbf{y})$ such that*

$$d\mathbb{G}(\mathbf{x}) = \chi^2 \bar{\mathbf{u}}_j(\mathbf{y}) \otimes \bar{\mathbf{l}}_j(\mathbf{x}) + \chi \bar{\bar{\mathbf{u}}}_j(\mathbf{y}) \otimes \bar{\bar{\mathbf{l}}}_j(\mathbf{x}) + O(\mu\chi).$$

Moreover there exist vector w_j, \tilde{w}_j and linear functionals $\bar{\mathbf{l}}_j, \bar{\bar{\mathbf{l}}}_j$ ($j = 1, 2$ meaning the first or second collision) such that if $\delta, \mu, \frac{1}{\chi} \rightarrow 0$ then

$$\bar{\mathbf{l}}_j(\mathbf{x}) \rightarrow \hat{\mathbf{l}}_j, \quad \bar{\bar{\mathbf{l}}}_j(\mathbf{x}) \rightarrow \hat{\bar{\mathbf{l}}}_j, \quad \text{and } \text{span}(\bar{\mathbf{u}}_j(\mathbf{y}), \bar{\bar{\mathbf{u}}}_j(\mathbf{y})) \rightarrow \text{span}(w_j, \tilde{w}_j),$$

$$\text{where } \bar{\mathbf{l}}_j = (1, 0_{1 \times 9}), \quad \bar{\bar{\mathbf{l}}}_j = - \left(\frac{\tilde{G}_{4,j}/\tilde{L}_{4,j}}{\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2}, 0_{1 \times 7}, -\frac{1}{\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2}, \frac{1}{\tilde{L}_{4,j}} \right),$$

$$\tilde{w} = (0, 1, 0_{1 \times 8})^T, \quad w_j = \left(0_{1 \times 8}; 1, -\frac{\hat{L}_{4,j}}{\hat{L}_{4,j}^2 + \hat{G}_{4,j}^2} \right)^T,$$

and $\tilde{L}_{4,j}, \tilde{G}_{4,j}$ stand for Gerver's values of $L_{4,j}, G_{4,j}$ after the j -th collision (initial value of \mathbb{G}) and $\hat{L}_{4,j}, \hat{G}_{4,j}$ stand for Gerver's values before the $(3-j)$ -th collision (final value of \mathbb{G}), which can be found in Appendix B.

Remark 3.1. The vectors $\hat{\mathbf{l}}, \hat{\mathbf{l}}, w, \tilde{w}$ agree with those in ([DX], equation (3.3)) if we eliminate the four 0 entries corresponding to x_1, v_1 .

The proof of Lemma 3.2 is a lengthy calculation based on Proposition 5.1. We put it in Appendix C.

3.2. The nondegeneracy condition.

Lemma 3.3. *The following non degeneracy conditions are satisfied.*

- (a1) $\text{span}(\hat{u}_1, B(\hat{\mathbf{l}}_1(\tilde{w})d\mathcal{R}w_2 - \hat{\mathbf{l}}_1(d\mathcal{R}w_2)\tilde{w}))$ is transversal to $\text{Ker}(\hat{\mathbf{l}}_1) \cap \text{Ker}(\hat{\mathbf{l}}_1)$.
- (a2) $de_4(d\mathcal{R}w_2) \neq 0$.
- (b1) $\text{span}(\hat{u}_2, B(\hat{\mathbf{l}}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}))$ is transversal to $\text{Ker}(\hat{\mathbf{l}}_2) \cap \text{Ker}(\hat{\mathbf{l}}_2)$.
- (b2) $de_4(w_1) \neq 0$.

The proof of this lemma is given in Section 3.4.

3.3. Proofs of Lemma 2.5, 2.6.

We now define the invariant cone fields.

Definition 3.1 (Invariant cone fields). *We now take \mathcal{K}_1 to be the set of vectors which make an angle less than a small constant η with $\text{span}(d\mathcal{R}w_2, \tilde{w}_2)$, and \mathcal{K}_2 to be the set of vectors which make an angle less than a small constant η with $\text{span}(w_1, \tilde{w}_1)$.*

Proof of Lemma 2.5. Consider for example the case where $\mathbf{x} \in U_2(\delta)$. We claim that if δ, μ are small enough then $d\mathbb{L}(\text{span}(w_1, \tilde{w}))$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\mathbf{l}}_2$. Indeed take Γ such that $\mathbf{l}_2(\Gamma) = 0$. If $\Gamma = aw_1 + \tilde{a}\tilde{w}$ then $a\mathbf{l}_2(w_1) + \tilde{a}\mathbf{l}_2(\tilde{w}) = 0$. It follows that the direction of Γ is close to the direction of $\hat{\Gamma} = \mathbf{l}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}$. Next take $\tilde{\Gamma} = bw_1 + \tilde{b}\tilde{w}$ where $b\mathbf{l}_2(w_1) + \tilde{b}\mathbf{l}_2(\tilde{w}) \neq 0$. Then the direction of $d\mathbb{L}\tilde{\Gamma}$ is close to \hat{u}_2 and the direction of $d\mathbb{L}(\Gamma)$ is close to $B(\hat{\Gamma})$ so our claim follows from Lemma 3.3.

Thus for any plane Π close to $\text{span}(w_1, \tilde{w})$ we have that $d\mathbb{L}(\Pi)$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\mathbf{l}}_2$. Take any $Y \in \mathcal{K}_2$. Then either Y and w_1 are linearly independent or Y and \tilde{w} are linearly independent. Hence $d\mathbb{L}(\text{span}(Y, w_1))$ or $d\mathbb{L}(\text{span}(Y, \tilde{w}))$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\mathbf{l}}_2$. Accordingly either $\mathbf{l}_2(d\mathbb{L}(Y)) \neq 0$ or $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) \neq 0$. If $\mathbf{l}_2(d\mathbb{L}(Y)) \neq 0$ then the direction of $d(\mathbb{G} \circ \mathbb{L})(Y)$ is close to $\bar{\mathbf{u}}$. If $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) \neq 0$ then the direction of $d(\mathbb{G} \circ \mathbb{L})(Y)$ is close to $\bar{\mathbf{u}}$. In either case $d(\mathbb{G} \circ \mathbb{L})(Y) \in \mathcal{K}_1$ and $\|d(\mathbb{G} \circ \mathbb{L})(Y)\| \geq c\chi\|Y\|$. This completes the proof in the case $\mathbf{x} \in U_2(\delta)$. The case where $\mathbf{x} \in U_1(\delta)$ is similar. \square

Proof of Lemma 2.6. Part (a) follows from the definition of \mathcal{K}_i . Also by part (b) of Lemma 3.3 the map $\pi : \text{span}(w, \tilde{w}) \rightarrow \mathbb{R}^2$ given by $\pi(\Gamma) = (d\ell_3(\Gamma), de_4(\Gamma))$ is

invertible. Namely if $\Gamma = aw + \tilde{a}\tilde{w}$ then

$$a = \frac{de_4(\Gamma)}{de_4(w)}, \quad \tilde{a} = d\ell_3(\Gamma) - ad\ell_3(w).$$

Accordingly π is invertible on planes close to $\text{span}(w, \tilde{w})$ proving our claim. \square

3.4. Checking the transversality.

Lemma 3.4. *The vectors \mathbf{l}, \mathbf{u} in the $O(1/\mu)$ part of the matrix $d\mathbb{L}$ satisfy the following:*

(a) *As $\mu \rightarrow 0$, we have*

$$\hat{\mathbf{l}}_j \cdot \tilde{w} \neq 0, \quad \hat{\mathbf{l}}_j \cdot w_{3-j} \neq 0, \quad \hat{\hat{\mathbf{l}}}_j \cdot \hat{\mathbf{u}}_j \neq 0,$$

$j = 1, 2$ meaning the first or the second collision.

(b) *If Q_3 and Q_4 switch roles after the collisions, the vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ get a “-” sign.*

The computation is done using the choice of $E_3^ = -\frac{1}{2}$ and $e_3^* = \frac{1}{2}$, at Gerver’s collision points.*

To check the nondegeneracy condition, it is enough to know the following.

Lemma 3.5. *Let $\mathbf{x} \in U_j(\delta)$ where δ is small enough. If we take the directional derivative at \mathbf{x} of the local map along a direction $\Gamma \in \text{span}\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\}$, such that*

$$\bar{\mathbf{l}}_j \cdot (d\mathbb{L}\Gamma) = 0, \quad j = 1, 2$$

then

$$\lim_{1/\chi \ll \mu \rightarrow 0} \frac{\partial E_3^+}{\partial \Gamma} \neq 0,$$

where E_3^+ is the energy of q_3 after the close encounter with q_4 . These derivatives are evaluated at Gerver’s collision points with $E_3^ = -1/2, e_3^* = 1/2$. See the Appendix B.2 for concrete values.*

The proofs of the two lemmas are postponed to Section 10.

Now we can check the nondegeneracy condition.

Proof of Lemma 3.3. We prove (b1) and (b2). The proofs of (a1) and (a2) are similar. To check (b2), de_4 we differentiate $e_4 = \sqrt{1 + (G_4/L_4)^2}$ to get

$$de_4 = \frac{1}{e_4} \left(\frac{G_4}{L_4^2} dG_4 - \frac{G_4^2}{L_4^3} dL_4 \right).$$

Thus Lemma 3.2 gives $de_4 w = \frac{G_4}{L_4^2} \neq 0$ as claimed.

Next we check (b1) which is equivalent to the following condition

$$(3.1) \quad \det \begin{pmatrix} \hat{\mathbf{l}}_2(\hat{\mathbf{u}}_2) & \hat{\mathbf{l}}_2(\hat{B}_2\Gamma') \\ \hat{\hat{\mathbf{l}}}_2(\hat{\mathbf{u}}_2) & \hat{\hat{\mathbf{l}}}_2(\hat{B}_2\Gamma') \end{pmatrix} \neq 0.$$

where $\Gamma' = \hat{\mathbf{l}}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}$. The vector $\Gamma' \neq 0$ due to part (a) of Lemma 3.4.

Let Γ be a vector satisfying $\hat{\mathbf{l}}_2 \cdot (d\mathbb{L}\Gamma) = 0$ and $d\mathbb{L}\Gamma$ is a vector in $\text{span}\{\hat{\mathbf{u}}_i, \hat{B}_i\Gamma'_i\}$, so it can be represented as $d\mathbb{L}\Gamma_i = b\hat{\mathbf{u}}_2 + b'\hat{B}_2\Gamma'$. Thus we can take $b = -\hat{\mathbf{l}}_2 \cdot \hat{B}_2\Gamma'$ and $b' = \hat{\mathbf{l}}_2 \cdot \hat{\mathbf{u}}_2$ to ensure that $d\mathbb{L}\Gamma_i \in \text{Ker}\hat{\mathbf{l}}_2$. Note that we have $b' \neq 0$ by part (a) of Lemma 3.4. Hence

$$\det \begin{pmatrix} \hat{\mathbf{l}}_2(\hat{\mathbf{u}}_2) & \hat{\mathbf{l}}_2(\hat{B}_2\Gamma') \\ \hat{\mathbf{l}}_2(\hat{\mathbf{u}}_2) & \hat{\mathbf{l}}_2(\hat{B}_2\Gamma') \end{pmatrix} = \frac{1}{b'} \det \begin{pmatrix} \hat{\mathbf{l}}_2(\hat{\mathbf{u}}_2) & \hat{\mathbf{l}}_2(d\mathbb{L}\Gamma) \\ \hat{\mathbf{l}}_2(\hat{\mathbf{u}}_2) & \hat{\mathbf{l}}_2(d\mathbb{L}\Gamma) \end{pmatrix} = \hat{\mathbf{l}}_2(d\mathbb{L}\Gamma)$$

where the last equality holds since $\hat{\mathbf{l}}_2(d\mathbb{L}\Gamma) = 0$. By Lemma 3.2, we have $\hat{\mathbf{l}}_i = (1, 0_{1 \times 9})$. Therefore $\hat{\mathbf{l}}_2(d\mathbb{L}\Gamma) = \frac{\partial E_3^+}{\partial \Gamma}$ and so (b2) follows from Lemma 3.5. \square

Remark 3.2. *Let us describe the physical and geometrical meanings of the vectors $\bar{\mathbf{l}}, \bar{\mathbf{l}}, \bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{l}, \mathbf{u}$ and the results in this section.*

- (1) *The structure of $d\mathbb{L}$ shows that a significant change of the behavior of the outgoing orbit parameters occurs when we vary the orbit parameters in the direction of \mathbf{l} , which is actually varying the closest distance (called impact parameter) between Q_3 and Q_4 (see Section 10, especially Corollary 10.1). The vector w in $d\mathbb{G}$ means that after the global map, the variable G_4 gets significant change as asserted by Lemma 2.7. So $\hat{\mathbf{l}}_i \cdot w_{3-i} \neq 0$ in Lemma 3.4 means that by changing G_4 after the global map, we can change the impact parameter and hence change the outgoing orbit parameters after the local map significantly. Similarly we see $\hat{\mathbf{l}}_i \cdot \tilde{w} \neq 0$ means the same outcome by varying ℓ_3 instead of G_4 .*
- (2) *The result $\hat{\mathbf{l}}_i \cdot \hat{\mathbf{u}}_i \neq 0$ in Lemma 3.4 means that by changing the outgoing orbit parameter of the local map in $\hat{\mathbf{u}}$ direction, which is in turn changed significantly by changing the impact parameter in the local map, we can change the final orbit parameter of the global map in the $\bar{\mathbf{u}}$ direction significantly. The vector $\hat{\mathbf{l}}$ has clear physical meaning. If we differentiate the outgoing asymptote $\theta_4^+ = g_4^+ - \arctan \frac{G_4^+}{L_4^+}$, where $+$ means after close encounter of Q_3 and Q_4 , we get $d\theta_4^+ = L_4^+ \hat{\mathbf{l}}$.*
- (3) *Lemma 3.5 means that if we vary the incoming orbit parameter of the local map in the direction Γ such that there is no significant change of the outgoing parameters of the local map in certain direction, then the energy (and, hence, semimajor axis) of the ellipse after Q_3, Q_4 interaction will change accordingly. One may think this as varying the incoming orbit parameter while holding the outgoing asymptotes unchanged. The change of energy means the change of periods of the ellipses according to Kepler's law. Ellipses with different periods will accumulate huge phase difference during one return time $O(\chi)$ of Q_4 . This is the mechanism that we use to fine tune the phase of Q_3 such that Q_3 comes to the correct phase to interact with Q_4 . Since the phase is defined up to 2π , we get a Cantor set as initial condition of singular orbits.*

4. SYMPLECTIC TRANSFORMATIONS AND POINCARÉ SECTIONS

In this section we define several Poincaré sections and perform symplectic transformations in the regions between the consecutive sections to make the Hamiltonian system suitable for doing calculations.

4.1. Three two-body problems. We start with the Hamiltonian (1.1). The translation invariance enables us to remove one body in the Hamiltonian. We choose Q_2 as this body. We start with the symplectic form

$$\begin{aligned}\omega &= \sum_{i=1}^4 dP_i \wedge dQ_i = d(P_1 + P_2 + P_3 + P_4) \wedge dQ_2 + dP_1 \wedge d(Q_1 - Q_2) \\ &\quad + dP_3 \wedge d(Q_3 - Q_2) + dP_4 \wedge d(Q_4 - Q_2) \\ &= d(P_1 + P_2 + P_3 + P_4) \wedge dQ_2 + dP_1 \wedge dq_1 + dP_3 \wedge dq_3 + dP_4 \wedge dq_4,\end{aligned}$$

where we have used (2.1). If we choose the mass center of the four bodies as the origin, then $P_1 + P_2 + P_3 + P_4 = 0$. Now the Hamiltonian becomes

$$\begin{aligned}H(q_1, P_1; q_3, P_3; q_4, P_4) &= \frac{1}{2}P_1^2 + \frac{1}{2}(P_1 + P_3 + P_4)^2 + \frac{1}{2\mu}P_3^2 + \frac{1}{2\mu}P_4^2 \\ &\quad - \frac{1}{|q_1|} - \frac{\mu}{|q_3|} - \frac{\mu}{|q_4|} - \frac{\mu}{|q_1 - q_3|} - \frac{\mu}{|q_1 - q_4|} - \frac{\mu^2}{|q_3 - q_4|} \\ &= P_1^2 + \frac{1}{2}\left(1 + \frac{1}{\mu}\right)(P_3^2 + P_4^2) + (\langle P_1, P_3 \rangle + \langle P_1, P_4 \rangle + \langle P_3, P_4 \rangle) \\ &\quad - \frac{1}{|q_1|} - \frac{\mu}{|q_3|} - \frac{\mu}{|q_4|} - \frac{\mu}{|q_1 - q_3|} - \frac{\mu}{|q_1 - q_4|} - \frac{\mu^2}{|q_3 - q_4|}.\end{aligned}$$

Now we restrict to the subspace where $P_1 + P_2 + P_3 + P_4 = 0$. Up to a factor μ , the restriction of the symplectic form ω is $\bar{\omega}$ defined in (2.4). Therefore we need to divide the whole Hamiltonian by μ to get

$$\begin{aligned}H(q_1, p_1; q_3, p_3; q_4, p_4) &= \mu p_1^2 + \frac{\mu}{2}\left(1 + \frac{1}{\mu}\right)(p_3^2 + p_4^2) + \mu(\langle p_1, p_3 \rangle + \langle p_1, p_4 \rangle + \langle p_3, p_4 \rangle) \\ &\quad - \frac{1}{\mu|q_1|} - \frac{1}{|q_3|} - \frac{1}{|q_4|} - \frac{1}{|q_1 - q_3|} - \frac{1}{|q_1 - q_4|} - \frac{\mu}{|q_3 - q_4|}.\end{aligned}$$

Remark 4.1. *In the new coordinates the total angular momentum equals to*

$$G = Q_1 \times P_1 - Q_2 \times (P_1 + P_3 + P_4) + Q_3 \times P_3 + Q_4 \times P_4 = q_1 \times P_1 + q_3 \times P_3 + q_4 \times P_4.$$

Therefore the angular momentum conservation takes form

$$\sum_{j=3,1,4} q_j \times p_j = \text{Const.}$$

4.2. More Poincaré sections. When Q_4 is closer to $Q_i, i = 1, 2$, we treat its motion as a hyperbolic Kepler motion with focus at Q_i and perturbed by Q_3, Q_{3-i} .

Definition 4.1. *We introduce one more set of coordinates.*

$$(4.2) \quad \begin{cases} v_3 = p_3 + \frac{\mu}{1+\mu}(p_1 + p_4), \\ v_1 = p_1 + p_4, \\ v_4 = \frac{1}{1+\mu}p_4 - \frac{\mu}{1+\mu}p_1, \end{cases} \quad \begin{cases} x_3 = q_3, \\ x_1 = \frac{1}{1+\mu}q_1 - \frac{\mu}{1+\mu}q_3 + \frac{\mu}{1+\mu}q_4, \\ x_4 = q_4 - q_1. \end{cases}$$

One can check that the transformation (4.2) is symplectic with respect to the symplectic form $\bar{\omega}$.

To distinguish these coordinates from those of Definition 2.1, we use superscript R (meaning *right*) and write $(x_3, v_3; x_1, v_1; x_4, v_4)^R = (x_3^R, v_3^R; x_1^R, v_1^R; x_4^R, v_4^R)$ for the coordinates from Definition 2.1 and use superscript L (meaning *left*) and write $(x_3, v_3; x_1, v_1; x_4, v_4)^L = (x_3^L, v_3^L; x_1^L, v_1^L; x_4^L, v_4^L)$ for the coordinates from Definition 4.1. Notice that $(x_3, v_3)^R = (x_3, v_3)^L$, so we omit this superscript for simplicity.

Definition 4.2 (Further Poincaré sections). *We further define two more sections to cut the global map into three pieces (see Figure 2).*

- Map (I) is the Poincaré map between the sections $\{x_{4,\parallel}^R = -2, v_{4,\parallel}^R < 0\}$ and $\{x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0\}$.
- Map (III) is the Poincaré map between the sections $\{x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0\}$ and $\{x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0\}$.
- Map (V) is the Poincaré map between the sections $\{x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0\}$ and $\{x_{4,\parallel}^R = -2, v_{4,\parallel}^R > 0\}$.
- We also introduce map (II) to change coordinates from right to the left on the section $\{x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0\}$ and map (IV) to change coordinates from left to the right on the section $\{x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0\}$.

Remark 4.2. *For most of the time Q_4 is moving between Q_1 and Q_2 , so that $q_{4,\parallel}$ is between 0 and $-\chi$. Hence $x_{4,\parallel}^R$ is between 0 and $-\chi$, and $x_{4,\parallel}^L$ is between 0 and χ for most of the time according to Definitions (4.2) and (2.3). The two sections $\{x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0\}$ and $\{x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0\}$ lie almost in the midway of Q_1 and Q_2 .*

In Section 4.3, 4.4, we will write the equations of motion as three Kepler motions $(x_i, v_i)^{R,L}$, $i = 3, 1, 4$ with perturbations. When perturbation is neglected $(x_4, v_4)^R$ is a hyperbola focused at origin and opening to the left while $(x_4, v_4)^L$ is a hyperbola focused at origin and opening to the right.

In the following subsections we describe the suitable changes of variables adapted to maps (I), (III), (V) as well as the local map \mathbb{L} .

4.3. The right case, when Q_4 is closer to Q_2 . We write the Hamiltonian in terms of three Kepler motions with perturbations.

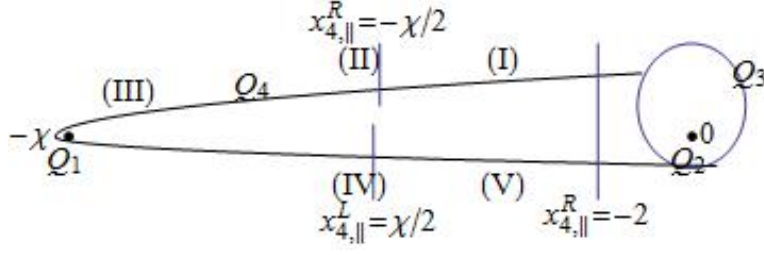


FIGURE 3. Poincaré sections

To remove the inner product terms in (4.1) we introduce new momenta

$$\begin{aligned}\bar{\omega} &= dp_1 \wedge d\left(q_1 - \frac{\mu(q_3 + q_4)}{2\mu + 1}\right) + d\left(p_3 + \frac{\mu}{1 + 2\mu}p_1\right) \wedge dq_3 + d\left(p_4 + \frac{\mu}{1 + 2\mu}p_1\right) \wedge dq_4 \\ &:= dp_1 \wedge dx_1 + d\tilde{p}_3 \wedge dq_3 + d\tilde{p}_4 \wedge dq_4.\end{aligned}$$

Then the Hamiltonian becomes

$$\begin{aligned}H(x_1, p_1; q_3, \tilde{p}_3; q_4, \tilde{p}_4) &= \frac{\mu(1 + \mu)}{1 + 2\mu}p_1^2 + \frac{\mu}{2}\left(1 + \frac{1}{\mu}\right)(\tilde{p}_3^2 + \tilde{p}_4^2) + \mu\langle\tilde{p}_3, \tilde{p}_4\rangle - \\ &\frac{1}{\mu\left|x_1 + \frac{\mu(q_3 + q_4)}{2\mu + 1}\right|} - \frac{1}{\left|x_1 + \frac{\mu(q_3 + q_4)}{2\mu + 1} - q_3\right|} - \frac{1}{\left|x_1 + \frac{\mu(q_3 + q_4)}{2\mu + 1} - q_4\right|} - \frac{1}{|q_3|} - \frac{1}{|q_4|} - \frac{\mu}{|q_3 - q_4|}.\end{aligned}$$

We perform one more symplectic change to kill $\langle\tilde{p}_3, \tilde{p}_4\rangle$

$$\begin{aligned}\bar{\omega} &= dp_1 \wedge dx_1 + d\left(\tilde{p}_3 + \frac{\mu\tilde{p}_4}{1 + \mu}\right) \wedge dq_3 + d\tilde{p}_4 \wedge d\left(q_4 - \frac{\mu q_3}{1 + \mu}\right) \\ &:= dv_1 \wedge dx_1 + dv_3 \wedge dx_3 + dv_4 \wedge dx_4.\end{aligned}$$

The composition of the two symplectic transformations gives (2.3). The Hamiltonian becomes

$$\begin{aligned}(4.3) \quad H(x_1, v_1; x_3, v_3; x_4, v_4) &= \frac{\mu(1 + \mu)}{1 + 2\mu}v_1^2 + \frac{1 + \mu}{2}v_3^2 + \frac{1 + 2\mu}{2(1 + \mu)}v_4^2 + \\ &\left(-\frac{1}{\left|x_4 + \frac{\mu x_3}{1 + \mu}\right|} - \frac{\mu}{\left|\frac{x_3}{1 + \mu} - x_4\right|}\right) - \frac{1}{|x_3|} + \left(-\frac{1}{\mu\left|x_1 + \frac{\mu}{1 + 2\mu}x_4 + \frac{\mu}{1 + \mu}x_3\right|} - \right. \\ &\left.\frac{1}{\left|x_1 + \frac{\mu}{1 + 2\mu}x_4 - \frac{1}{1 + \mu}x_3\right|} - \frac{1}{\left|x_1 - \frac{1 + \mu}{1 + 2\mu}x_4\right|}\right) \\ &= \frac{\mu(1 + \mu)}{1 + 2\mu}v_1^2 + \frac{1 + \mu}{2}v_3^2 + \frac{1 + 2\mu}{2(1 + \mu)}v_4^2 - \left[\frac{2\mu + 1}{\mu|x_1|} + \frac{1}{|x_3|} + \frac{1 + \mu}{|x_4|}\right] + U^R,\end{aligned}$$

where

$$(4.4) \quad U^R = \frac{2\mu + 1}{\mu|x_1|} - \left(\frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu}x_4 + \frac{\mu}{1+\mu}x_3 \right|} + \frac{1}{\left| x_1 + \frac{\mu}{1+2\mu}x_4 - \frac{1}{1+\mu}x_3 \right|} + \frac{1}{\left| x_1 - \frac{1+\mu}{1+2\mu}x_4 \right|} \right) + \frac{1+\mu}{|x_4|} - \left(\frac{1}{\left| x_4 + \frac{\mu x_3}{1+\mu} \right|} + \frac{\mu}{\left| \frac{x_3}{1+\mu} - x_4 \right|} \right).$$

We choose to include the potentials $\left[\frac{2\mu + 1}{\mu|x_1|} + \frac{1}{|x_3|} + \frac{1+\mu}{|x_4|} \right]$ in (4.3) to the unperturbed part so that U^R is a controllable perturbation to the Kepler motions after some cancelations. We will study U^R in more details in Lemma 6.1.

Using the Appendix A to convert x_3 and x_4 to the Delaunay variables we finally get

$$H(L_3, \ell_3, G_3, g_3; x_1, v_1; L_4, \ell_4, G_4, g_4) = \left(\frac{v_1^2}{2m_{1R}} - \frac{k_{1R}}{|x_1|} \right) - \frac{m_{3R}k_{3R}^2}{2L_3^2} + \frac{m_{4R}k_{4R}^2}{2L_4^2} + U^R$$

where

$$m_{1R} = \frac{1+2\mu}{2\mu(1+\mu)}, \quad m_{3R} = \frac{1}{1+\mu}, \quad m_{4R} = \frac{1+\mu}{1+2\mu}, \quad k_{1R} = \frac{1+2\mu}{\mu}, \quad k_{3R} = 1, \quad k_{4R} = 1+\mu.$$

Remark 4.3. Note that the total angular momentum conservation takes form $G_1 + G_3 + G_4 = \text{const}$, where $G_i = v_i \times x_i$, $i = 3, 1, 4$. Indeed,

$$\sum_{i=3,1,4} v_i \times x_i = \sum_{i=3,1,4} p_i \times q_i$$

since the transformation (2.3) is linear symplectic.

4.4. The left case, when Q_4 is closer to Q_1 . In this section, we explain the choice of (4.2) and derive the corresponding Hamiltonian. When Q_4 is moving between the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$ and turns around Q_1 , we treat Q_4 's motion as an approximate hyperbola with focus at Q_1 . So we do the following change of variables

$$\begin{aligned} \bar{\omega} &= dp_1 \wedge dq_1 + dp_3 \wedge dq_3 + dp_4 \wedge dq_4 = d(p_1 + p_4) \wedge dq_1 + dp_4 \wedge d(q_4 - q_1) + dp_3 \wedge dq_3 \\ &:= dp_{14} \wedge dq_1 + dp_4 \wedge dq_{14} + dp_3 \wedge dq_3. \end{aligned}$$

Now the Hamiltonian is

$$\begin{aligned} H(q_1, p_{14}; q_{14}, p_4; q_3, p_3) &= \mu p_{14}^2 + \frac{\mu}{2} \left(1 + \frac{1}{\mu} \right) (p_3^2 + p_4^2) + \mu (\langle p_{14}, p_3 \rangle - \langle p_{14}, p_4 \rangle) \\ &\quad - \frac{1}{\mu|q_1|} - \frac{1}{|q_3|} - \frac{1}{|q_{14} + q_1|} - \frac{1}{|q_1 - q_3|} - \frac{1}{|q_{14}|} - \frac{\mu}{|q_3 - q_{14} - q_1|}. \end{aligned}$$

The next symplectic transformation is

$$\begin{aligned} \bar{\omega} &= dp_{14} \wedge d \left(q_1 - \frac{\mu(q_3 - q_{14})}{1+\mu} \right) + d \left(p_4 - \frac{\mu p_{14}}{1+\mu} \right) \wedge dq_{14} + d \left(p_3 + \frac{\mu p_{14}}{1+\mu} \right) \wedge dq_3 \\ &:= dv_1 \wedge dx_1 + dv_4 \wedge dx_4 + dv_3 \wedge dx_3. \end{aligned}$$

The composition of the two symplectic transformations gives (4.2). The Hamiltonian becomes

$$\begin{aligned}
 H(x_1, v_1; x_3, v_4; x_4, v_4) &= \frac{\mu}{1+\mu} v_1^2 + \frac{\mu}{2} \left(1 + \frac{1}{\mu}\right) (v_3^2 + v_4^2) \\
 &\quad - \frac{1}{|x_3|} - \frac{1}{|x_4|} - \left(\frac{1}{\mu \left| x_1 + \frac{\mu}{1+\mu} x_3 - \frac{\mu}{1+\mu} x_4 \right|} + \frac{1}{\left| x_1 + \frac{\mu}{1+\mu} x_3 + \frac{1}{1+\mu} x_4 \right|} \right. \\
 (4.5) \quad &\quad \left. + \frac{1}{\left| x_1 - \frac{1}{1+\mu} x_3 - \frac{\mu}{1+\mu} x_4 \right|} + \frac{\mu}{\left| x_1 - \frac{1}{1+\mu} x_3 + \frac{1}{1+\mu} x_4 \right|} \right) \\
 &= \frac{\mu}{1+\mu} v_1^2 + \frac{\mu}{2} \left(1 + \frac{1}{\mu}\right) (v_3^2 + v_4^2) - \frac{(1+\mu)^2}{\mu |x_1|} - \frac{1}{|x_3|} - \frac{1}{|x_4|} + U^L
 \end{aligned}$$

where

$$\begin{aligned}
 U^L &= \frac{(1+\mu)^2}{\mu |x_1|} - \left(\frac{1}{\mu \left| x_1 + \frac{\mu}{1+\mu} x_3 - \frac{\mu}{1+\mu} x_4 \right|} + \frac{1}{\left| x_1 + \frac{\mu}{1+\mu} x_3 + \frac{1}{1+\mu} x_4 \right|} + \right. \\
 (4.6) \quad &\quad \left. \frac{1}{\left| x_1 - \frac{1}{1+\mu} x_3 - \frac{\mu}{1+\mu} x_4 \right|} + \frac{\mu}{\left| x_1 - \frac{1}{1+\mu} x_3 + \frac{1}{1+\mu} x_4 \right|} \right).
 \end{aligned}$$

Some remarkable cancelations occur in U^L so that it is actually a controllable perturbation. See Lemma 6.1 for more details.

Finally, we use the Appendix A to transform Q_3 and Q_4 variables to the Delaunay coordinates obtaining

$$H(L_3, \ell_3, G_3, g_3; x_1, v_1; L_4, \ell_4, G_4, g_4) = \left(\frac{v_1^2}{2m_{1L}} - \frac{k_{1L}}{|x_1|} \right) - \frac{m_{3L} k_{3L}^2}{2L_3^2} + \frac{m_{4L} k_{4L}^2}{2L_4^2} + U^L,$$

$$\text{where } m_{1L} = \frac{1+\mu}{2\mu}, \quad m_{3L} = m_{4L} = \frac{1}{1+\mu}, \quad k_{1L} = \frac{(1+\mu)^2}{\mu}, \quad k_{3L} = k_{4L} = 1.$$

4.5. Local map, away from close encounter. We cut the local map into three pieces by introducing a new section $|q_3 - q_4| = \mu^\kappa$, $1/3 < \kappa < 1/2$. The restriction $\kappa < 1/2$ comes from the proof Lemma 10.2 in Section 7 where we need $\mu^{1-2\kappa}$ to be small, and the restriction $\kappa > 1/3$ comes from the proof of Lemma 10.1 and (10.25) where we need $\mu^{3\kappa-1}$ to be small.

When Q_3, Q_4 are moving outside the sphere $|q_3 - q_4| = \mu^\kappa$, we use the same transformation as (2.3) but different ways of grouping terms. So we get the following from the first equality of (4.3)

(4.7)

$$\begin{aligned}
H((x_1, v_1; x_3, v_3; x_4, v_4)^R) &= \frac{\mu(1+\mu)}{1+2\mu} v_1^2 + \frac{1+\mu}{2} v_3^2 + \frac{1+2\mu}{2(1+\mu)} v_4^2 - \\
&\left[\frac{2\mu+1}{\mu|x_1|} + \frac{1}{|x_3|} + \frac{1}{|x_4|} \right] - \frac{\mu}{\left| \frac{x_3}{1+\mu} - x_4 \right|} + \left(\frac{1}{|x_4|} - \frac{1}{\left| x_4 + \frac{\mu x_3}{1+\mu} \right|} \right) + \left(\frac{2\mu+1}{\mu|x_1|} - \right. \\
&\left. \frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} - \frac{1}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 - \frac{1}{1+\mu} x_3 \right|} - \frac{1}{\left| x_1 - \frac{1+\mu}{1+2\mu} x_4 \right|} \right) \\
&= \left(\frac{v_1^2}{2m_{1R}} - \frac{k_{1R}}{|x_1|} \right) - \frac{m_{3R}}{2L_3^2} + \frac{m_{4R}}{2L_4^2} - \frac{\mu}{\left| \frac{x_3}{1+\mu} - x_4 \right|} + \left[\left(\frac{1}{|x_4|} - \frac{1}{\left| x_4 + \frac{\mu x_3}{1+\mu} \right|} \right) + \right. \\
&\left. \left(\frac{2\mu+1}{\mu|x_1|} - \frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} - \frac{1}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 - \frac{1}{1+\mu} x_3 \right|} - \frac{1}{\left| x_1 - \frac{1+\mu}{1+2\mu} x_4 \right|} \right) \right].
\end{aligned}$$

We claim that if $|x_1| = O(\chi)$, and $|x_3|, |x_4|$ are bounded, then the perturbation in the bracket is estimated as $O(1/\chi^3)$.

To this end we use the following power series expansions for the potential. Let a and b be two vectors such that $|b| < |a|$. We use the identity

$$(4.8) \quad \frac{1}{|a+b|} = \frac{1}{|a|} \frac{1}{\sqrt{1 + 2\frac{\langle a, b \rangle}{|a|^2} + \frac{|b|^2}{|a|^2}}}$$

and then take the power series expansion of $(1+z)^{-1/2}$ at $z=0$. For the case at hand we use this formula with $a = x_1$ and b being the remaining terms. Note that the first parenthesis in bracket is $O(\mu)$. Next we split

$$\frac{2\mu+1}{\mu|x_1|} = \frac{1}{\mu|x_1|} + \frac{2}{|x_1|}.$$

Now we get using (4.8)

$$\begin{aligned}
\frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} &= \frac{1}{\mu|x_1|} - \frac{1}{|x_1|^3} (\langle x_4, x_1 \rangle + \langle x_3, x_1 \rangle) + O\left(\frac{1}{\chi^3}\right), \\
\frac{1}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 - \frac{1}{1+\mu} x_3 \right|} &= \frac{1}{|x_1|} + \frac{\langle x_3, x_1 \rangle}{|x_1|^3} + O\left(\frac{1}{\chi^3}\right), \\
\frac{1}{\left| x_1 - \frac{1+\mu}{1+2\mu} x_4 \right|} &= \frac{1}{|x_1|} + \frac{\langle x_4, x_1 \rangle}{|x_1|^3} + O\left(\frac{1}{\chi^3}\right).
\end{aligned}$$

The $1/|x_1|$ and inner product terms cancel leaving us with $O\left(\frac{1}{\chi^3}\right)$ remainder.

4.6. Local map, close encounter. When Q_3, Q_4 are moving inside the sphere $|q_3 - q_4| = \mu^\kappa$, we derive the Hamiltonian system describing the relative motion of

Q_3, Q_4 . We start with the Hamiltonian (4.1) and make the following symplectic changes to convert to the coordinates of relative motion and motion of mass center

$$(4.9) \quad \begin{cases} q_- = \frac{1}{2}(q_3 - q_4) \\ q_+ = \frac{1}{2}(q_3 + q_4) \\ q_1 = q_1 \end{cases}, \quad \begin{cases} p_- = p_3 - p_4 \\ p_+ = p_3 + p_4 \\ p_1 = p_1 \end{cases}.$$

The symplectic form is now

$$\bar{\omega} = dp_1 \wedge dq_1 + dp_+ \wedge dq_+ + dp_- \wedge dq_-.$$

Using (4.8) we see that the Hamiltonian (4.1) becomes

$$(4.10) \quad \begin{aligned} H(q_1, p_1; q_-, p_-; q_+, p_+) &= \left(\mu p_1^2 - \frac{1}{\mu |q_1|} \right) + \frac{1+2\mu}{4} p_+^2 + \left(\frac{1}{4} p_-^2 - \frac{\mu}{2|q_-|} \right) - \\ &\quad \mu \langle p_1, p_+ \rangle - \frac{1}{|q_+ - q_-|} - \frac{1}{|q_+ + q_-|} - \frac{1}{|q_1 - q_+ + q_-|} - \frac{1}{|q_1 - q_+ - q_-|} \\ &= \left(\mu p_1^2 - \frac{1+2\mu}{\mu |q_1|} \right) + \left(\frac{1+2\mu}{4} p_+^2 - \frac{2}{|q_+|} \right) + \left(\frac{1}{4} p_-^2 - \frac{\mu}{2|q_-|} \right) + \mu \langle p_1, p_+ \rangle - \\ &\quad \frac{3\langle q_+, q_- \rangle^2}{2|q_+|^5} + \frac{|q_-|^2}{|q_+|^3} - \frac{1}{|q_1|^3} \left(\langle q_1, q_+ \rangle - |q_+|^2 - |q_-|^2 + \frac{3}{2} \left(\frac{\langle q_1, q_+ \rangle^2 + \langle q_1, q_- \rangle^2}{|q_1|^2} \right) \right) \\ &\quad + O\left(|q_-|^3 + \frac{1}{|q_1|^3}\right) \\ &= \left(\mu p_1^2 - \frac{1+2\mu}{\mu |q_1|} \right) + \left(\frac{1+2\mu}{4} p_+^2 - \frac{2}{|q_+|} \right) + \frac{\mu^2}{4L_-^2} + \mu \langle p_1, p_+ \rangle + \frac{|q_-|^2}{|q_+|^3} - \frac{3\langle q_+, q_- \rangle^2}{2|q_+|^5} \\ &\quad - \frac{1}{|q_1|^3} \left(\langle q_1, q_+ \rangle - |q_+|^2 - |q_-|^2 + \frac{3}{2} \left(\frac{\langle q_1, q_+ \rangle^2 + \langle q_1, q_- \rangle^2}{|q_1|^2} \right) \right) + O(|q_-|^3 + 1/|q_1|^3). \end{aligned}$$

In the above derivation, we treat q_- as a small quantity to do the Taylor expansion, since $|q_-| \leq \mu^\kappa$ inside the sphere $|q_-| = 2\mu^\kappa$.

5. STATEMENT OF THE MAIN TECHNICAL PROPOSITION

In this section, we give the statement of our calculation of matrices needed in the proof of the global map. We use the coordinates system $(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4, g_4)$ to do the calculation. In the following, the superscript i means “initial” and f means “final”.

Notation 5.1. *To avoid many O notations in our estimates, we introduce the following conventions.*

- We use the notation $a \lesssim b$ if $a = O(b)$ or equivalently $|a| \leq C|b|$ for some constant C independent of χ, μ , and the notation $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$ hold.
- We also generalize this notation to vectors and matrices. For two vectors $A, B \in \mathbb{R}^n$, we write $A \lesssim B$ if $A_i \lesssim B_i$ holds for each entries A_i, B_i of A and B respectively, and write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$ hold. Similar for matrices.

- For a matrix $[\#]$, we refer to its blocks as $\begin{bmatrix} \#_{33} & \#_{31} & \#_{34} \\ \#_{13} & \#_{11} & \#_{14} \\ \#_{43} & \#_{41} & \#_{44} \end{bmatrix}$, and its (i, j) -th entry as $[\#](i, j)$, $i, j = 1, 2, \dots, 10$.

Moreover, when we use “ \lesssim ”, there are some entries in the vector or matrix, for which we have estimate in the sense of \sim . Those entries will be important to show that the χ^2 and χ terms in Lemma 3.2 do not vanish. For those entries, we use **bold font**.

Proposition 5.1. *Under the assumption **AG**, we have the following (a)*

- (a.1) *The derivative of the global map is a product of five 10×10 matrices $d\mathbb{G} = (V)(IV)(III)(II)(I)$ having the following form.*

$$\begin{aligned} (I) &= (\text{Id}_{10} + \chi u_1^f \otimes l_1^f) N_1 (\text{Id}_{10} + u_1^i \otimes l_1^i) \lesssim (\text{Id}_{10} + \chi u \otimes l) N_1 (\text{Id}_{10} + u_1^i \otimes l_1^i), \\ (II) &= (\chi u_{iii} \otimes l_{iii} + A) L \cdot R^{-1} (\chi u_i \otimes l_i + C), \\ (III) &= (\text{Id}_{10} + \chi u_3^f \otimes l_3^f) M (\text{Id}_{10} + \chi u_3^i \otimes l_3^i) \lesssim (\text{Id}_{10} + \chi u \otimes l) M (\text{Id}_{10} + \chi u \otimes l'), \\ (IV) &= (\chi u_{iii'} \otimes l_{iii'} + A) R \cdot L^{-1} (\chi u_{i'} \otimes l_{i'} + C) \\ (V) &= (\text{Id}_{10} + u_5^f \otimes l_5^f) N_5 (\text{Id}_{10} + \chi u_5^i \otimes l_5^i) \lesssim (\text{Id}_{10} + u_1^i \otimes l_1^i) N_5 (\text{Id}_{10} + \chi u \otimes l'), \end{aligned}$$

- (a.2) *where*

$$\begin{aligned} u_1^f, u_3^f, u_3^i, u_5^i &\lesssim u := \left(\frac{1}{\chi^3}, \mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2} \right)_{10 \times 1}^T, \\ l_1^f, l_3^f &\lesssim l := \left(\mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; \frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi} \right)_{1 \times 10}, \\ l_3^i, l_5^i &\lesssim l' := \left(\mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; \frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi} \right)_{1 \times 10}, \\ u_{iii}, u_{iii'} &\sim (0_{1 \times 8}; \mathbf{1}, \mathbf{1})_{10 \times 1}^T, \quad l_{iii}, l_{iii'} \lesssim \left(0_{1 \times 8}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mathbf{1}}{\chi}, \frac{\mu \mathcal{G}}{\chi}, \mathbf{1} \right)_{1 \times 12}, \\ u_i, u_{i'} &= \left(0_{1 \times 9}, -\frac{L_4}{m_4^2 k_4^2}, 0, \frac{\mathbf{1}}{\chi} \right)_{12 \times 1}^T, \\ l_i &\lesssim \left(\mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; \mathbf{1}, \mathbf{1} \right)_{1 \times 10}, \\ l_{i'} &\lesssim \left(\frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi^4}, \frac{\mu \mathcal{G}}{\chi^4}, \frac{\mu \mathcal{G}}{\chi^4}; \frac{\mathcal{G}}{\chi^3}, \frac{\mu \mathcal{G}^2}{\chi^4}, \frac{\mu^2 \mathcal{G}}{\chi}, \frac{\mu^2 \mathcal{G}^2}{\chi^2}; \mathbf{1}, \mathbf{1} \right)_{1 \times 10}, \\ u_5^f, u_1^i &\lesssim \left(\mu, 1, \mu, \mu; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \mu, \mu \right)_{10 \times 1}^T, \\ l_5^f, l_1^i &\lesssim \left(1, \mu, \mu, \mu; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; 1, 1 \right)_{1 \times 10}, \\ \text{(a.3) and } A &= \left[\begin{array}{c|c|c|c|c} \text{Id}_{8 \times 8} & 0_{8 \times 1} & 0_{8 \times 1} & 0_{8 \times 1} & 0_{8 \times 1} \\ \hline 0_{1 \times 8} & 0 & 0 & 0 & 0 \\ \hline 0_{1 \times 8} & O\left(\frac{1}{\chi^2}\right) & O\left(\frac{1}{\chi^2}\right) & O(1) & O(1) \end{array} \right]_{10 \times 12}, \end{aligned}$$

$$C = \left[\begin{array}{c|c} \text{Id}_{8 \times 8} & 0_{8 \times 2} \\ \hline 0_{1 \times 8} & 0_{1 \times 2} \\ 0_{1 \times 8} & O(1)_{1 \times 2} \\ \check{l}_i & O\left(\frac{\mu \mathcal{G}}{\chi}\right)_{1 \times 2} \\ 0_{1 \times 8} & 0_{1 \times 2} \end{array} \right]_{12 \times 10} \quad \text{with } \check{l}_i \lesssim \left(1, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}\right)_{1 \times 8}.$$

(a.4) Matrices R and L are the derivative matrices of the transformations (2.3) and (4.2) respectively, and they have the following expression in the coordinates $(x_3, v_3; x_1, v_1; x_4, v_4)$

$$(5.1) \quad R \cdot L^{-1}, L \cdot R^{-1} = \begin{bmatrix} \text{Id}_4 & 0 & 0 & 0 & 0 \\ 0 & m_{\pm} \text{Id}_2 & 0 & \mp \frac{2\mu}{1+2\mu} \text{Id}_2 & 0 \\ 0 & 0 & m_{\mp} \text{Id}_2 & 0 & \mp \text{Id}_2 \\ 0 & \pm \text{Id}_2 & 0 & m_{\mp} \text{Id}_2 & 0 \\ 0 & 0 & \pm \frac{2\mu}{1+2\mu} \text{Id}_2 & 0 & m_{\pm} \text{Id}_2 \end{bmatrix}_{12 \times 12},$$

where $m_+ = \frac{1+\mu}{1+2\mu}$ and $m_- = \frac{1}{1+\mu}$, and we choose the upper sign for $R \cdot L^{-1}$ and the lower sign for $L \cdot R^{-1}$ when we need to make a choice in \pm or \mp .

(a.5) The following estimates hold.

$$N_1 - \text{Id}_{10} \lesssim \left[\begin{array}{cc|cccc|c} \mu & \mu_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu^2}{\chi} & \mu^2 & \frac{\mu^2 \mathcal{G}}{\chi} & \mu_{1 \times 2} \\ \mu \chi & (\mu^2 \chi)_{1 \times 3} & \frac{1}{\mu \chi} & \frac{\mu}{\chi} & \mu \chi & \mu \mathcal{G} & (\mu^2 \chi)_{1 \times 2} \\ \mu & \mu_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu^2}{\chi} & \mu^2 & \frac{\mu^2 \mathcal{G}}{\chi} & \mu_{1 \times 2} \\ \mu & \mu_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu}{\chi} & \mu^2 & \frac{\mu^2 \mathcal{G}}{\chi} & \mu_{1 \times 2} \\ \hline \mu \chi & \mu^2 \chi_{1 \times 3} & \frac{1}{\chi} & \frac{\mu^2}{\chi} & \mu \chi & \mu^2 \mathcal{G} & (\mu^2 \chi)_{1 \times 2} \\ \mu \mathcal{G} & \mu^2 \mathcal{G}_{1 \times 3} & \frac{1}{\chi^2} & \frac{1}{\chi} & \mu^2 \mathcal{G} & \mu \chi & (\mu^2 \mathcal{G})_{1 \times 2} \\ \frac{1}{\mu \chi} & \left(\frac{1}{\chi}\right)_{1 \times 3} & \frac{1}{\mu \chi^2} & \frac{\mu}{\chi^2} & \frac{1}{\chi} & \frac{\mathcal{G}}{\chi^2} & \left(\frac{1}{\chi}\right)_{1 \times 2} \\ \frac{1}{\chi} & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^3} & \frac{1}{\mu \chi^2} & \frac{\mu}{\chi} & \frac{1}{\chi} & \left(\frac{\mu}{\chi}\right)_{1 \times 2} \\ \hline 1 & \mu_{1 \times 3} & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{\mu}{\chi} & \mu & \frac{\mu \mathcal{G}}{\chi} & \mathbf{1}_{1 \times 2} \\ 1 & \mu_{1 \times 3} & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{\mu}{\chi} & \mu & \frac{\mu \mathcal{G}}{\chi} & \mathbf{1}_{1 \times 2} \end{array} \right]_{10 \times 10}$$

$$M - \text{Id}_{10} \lesssim \left[\begin{array}{cc|cccc|c} \frac{\mu}{\chi} & \left(\frac{1}{\chi^2}\right)_{1 \times 3} & \frac{1}{\mu \chi^3} & \frac{1}{\chi^3} & \frac{\mu}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \left(\frac{1}{\chi^2}\right)_{1 \times 2} \\ \mu \chi & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{1}{\mu \chi} & \frac{1}{\chi^2} & \mu \chi & \mu \mathcal{G} & \mathbf{1}_{1 \times 2} \\ \frac{\mu}{\chi} & \left(\frac{1}{\chi^2}\right)_{1 \times 3} & \frac{1}{\mu \chi^3} & \frac{1}{\chi^3} & \frac{\mu}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \left(\frac{1}{\chi^2}\right)_{1 \times 2} \\ \frac{\mu}{\chi} & \left(\frac{1}{\chi^2}\right)_{1 \times 3} & \frac{1}{\mu \chi^3} & \frac{1}{\chi^3} & \frac{\mu}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \left(\frac{1}{\chi^2}\right)_{1 \times 2} \\ \hline \mu \chi & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{1}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \mu \chi & \mu^2 \mathcal{G} & (\mu)_{1 \times 2} \\ \mu \mathcal{G} & \left(\frac{\mu \mathcal{G}}{\chi^2}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^2} & \frac{1}{\chi} & \mu^2 \mathcal{G} & \mu \chi & \left(\frac{\mu \mathcal{G}}{\chi}\right)_{1 \times 2} \\ \frac{1}{\mu \chi} & \left(\frac{1}{\chi^3}\right)_{1 \times 3} & \frac{1}{\mu \chi^2} & \frac{1}{\chi^3} & \frac{1}{\chi} & \frac{\mathcal{G}}{\chi^2} & \left(\frac{\mu \mathcal{G}}{\chi^2}\right)_{1 \times 2} \\ \frac{1}{\chi} & \left(\frac{1}{\chi^3}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^3} & \frac{1}{\mu \chi^2} & \frac{\mu}{\chi} & \frac{1}{\chi} & \left(\frac{\mu \mathcal{G}}{\chi^2}\right)_{1 \times 2} \\ \hline 1 & \left(\frac{1}{\chi^2}\right)_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu \mathcal{G}}{\chi^2} & \mu & \frac{\mu \mathcal{G}}{\chi} & \mathbf{1}_{1 \times 2} \\ 1 & \left(\frac{1}{\chi^2}\right)_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu \mathcal{G}}{\chi^2} & \mu & \frac{\mu \mathcal{G}}{\chi} & \mathbf{1}_{1 \times 2} \end{array} \right]_{10 \times 10},$$

$$N_5 - \text{Id}_{10} \lesssim \begin{bmatrix} \mu^2\chi & \mu_{1 \times 3} & \frac{1}{\chi} & \frac{\mu^2}{\chi^2} & \mu^2\chi & \mu^2\mathcal{G} & \mu_{1 \times 2} \\ \mu\chi & \mu_{1 \times 3} & \frac{1}{\mu\chi} & \frac{\mu^2}{\chi^2} & \mu\chi & \mu\mathcal{G} & 1_{1 \times 2} \\ \mu^2\chi & \mu_{1 \times 3} & \frac{1}{\chi} & \frac{\mu^2}{\chi^2} & \mu^2\chi & \mu^2\mathcal{G} & \mu_{1 \times 2} \\ \mu^2\chi & \mu_{1 \times 3} & \frac{1}{\chi} & \frac{\mu^2}{\chi^2} & \mu^2\chi & \mu^2\mathcal{G} & \mu_{1 \times 2} \\ \mu\chi & (\mu^2)_{1 \times 3} & \frac{1}{\chi} & \frac{\mu^3}{\chi} & \mu\chi & \mu^2\mathcal{G} & (\mu)_{1 \times 2} \\ \mu\mathcal{G} & \left(\frac{\mu^2\mathcal{G}}{\chi}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^2} & \frac{1}{\chi} & \mu^2\mathcal{G} & \mu\chi & \left(\frac{\mu\mathcal{G}}{\chi}\right)_{1 \times 2} \\ \mu^2 & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{1}{\mu\chi^2} & \frac{\mu^2}{\chi^2} & \mu^2 & \frac{\mu^2\mathcal{G}}{\chi} & \left(\frac{\mu}{\chi}\right)_{1 \times 2} \\ \mu^2 & \left(\frac{\mu}{\chi}\right)_{1 \times 3} & \frac{1}{\chi^2} & \frac{1}{\mu\chi^2} & \mu^2 & \frac{\mu^2\mathcal{G}}{\chi} & \left(\frac{\mu}{\chi}\right)_{1 \times 2} \\ \mu^2\chi & \mu_{1 \times 3} & \frac{1}{\chi} & \frac{\mu}{\chi} & \mu^2\chi & \mu^2\mathcal{G} & 1_{1 \times 2} \\ \mu^2\chi & \mu_{1 \times 3} & \frac{1}{\chi} & \frac{\mu}{\chi} & \mu^2\chi & \mu^2\mathcal{G} & 1_{1 \times 2} \end{bmatrix}_{10 \times 10}.$$

(b) Moreover, for the $\mathbf{1}$ entries in (a2), we have the following exact estimates

(b.1) As $1/\chi \ll \mu \rightarrow 0$, we have

$$\begin{aligned} u_{1,3}^f, u_{3,5}^i, u &\rightarrow (0, 1, 0_{1 \times 8})_{10 \times 1}^T = \tilde{w}, \\ u_{iii} &\rightarrow \left(0_{1 \times 8}; 1, \frac{1}{\tilde{L}_{4,j}}\right)_{10 \times 1}^T, \quad u_{iii'} \rightarrow \left(0_{1 \times 8}; 1, -\frac{\hat{L}_{4,j}}{\hat{G}_{4,j}^2 + \hat{L}_{4,j}^2}\right)_{10 \times 1}^T = w_j, \\ l_{1,3}^f, l_{3,5}^i, l, l' &\rightarrow (1, 0_{1 \times 9})_{1 \times 10} = \hat{\mathbf{l}}_j, \\ l_i &\rightarrow -\left(\frac{\tilde{G}_{4,j}/\tilde{L}_{4,j}}{\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2}, 0_{1 \times 7}, -\frac{1}{\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2}, \frac{1}{\tilde{L}_{4,j}}\right)_{1 \times 10} = \hat{\mathbf{l}}_j. \end{aligned}$$

Here $j = 1, 2$ means the first and second collisions in Gerver's construction. \tilde{L}_3 and \tilde{G}_3 are the values of the Delaunay coordinates at the initial point for the global map and \hat{L}_3 and \hat{G}_3 are the values of the Delaunay coordinates at the final point. See also the statement of Lemma 3.2 for the convention of $\tilde{L}, \tilde{G}, \hat{L}, \hat{G}$.

(b.2) In addition, we have as $1/\chi \ll \mu \rightarrow 0$,

$$\begin{aligned} l_{i'} &\rightarrow \left(0_{1 \times 8}, \frac{1}{\tilde{L}_{4,j}^2}, -\frac{1}{\tilde{L}_{4,j}}\right)_{1 \times 10}, \quad u_i, u_{i'} = \left(0_{1 \times 9}, \frac{L_4}{2m_4^2 k_4^2}, 0, \frac{1}{\chi}\right)_{12 \times 1}^T, \\ l_{iii} &= -l_{iii'} = \left(0_{1 \times 8}; O\left(\frac{\mu\mathcal{G}}{\chi^2}\right), -\frac{m_4 k_4}{\chi L_4}, O\left(\frac{\mu\mathcal{G}}{\chi}\right), -\frac{1}{2}\right)_{1 \times 12}, \end{aligned}$$

(b.3) The $O(1)$ blocks in N_1, M, N_5 have exact estimates as follows,

$$\begin{aligned} (N_1)_{44} &\simeq \begin{bmatrix} 1 + \frac{\tilde{L}_{4,j}^2}{2(\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2)} & -\frac{\tilde{L}_{4,j}}{2} \\ \frac{\tilde{L}_{4,j}^3}{2(\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2)^2} & 1 - \frac{\tilde{L}_{4,j}^2}{2(\tilde{L}_{4,j}^2 + \tilde{G}_{4,j}^2)} \end{bmatrix}, \quad (M)_{44} \simeq \begin{bmatrix} \frac{1}{2} & -\frac{L_{4,j}}{2} \\ \frac{3}{2L_{4,j}} & \frac{1}{2} \end{bmatrix}, \\ (N_5)_{44} &\simeq \begin{bmatrix} 1 - \frac{1/2\hat{L}_{4,j}^2}{\hat{L}_4^2 + \hat{G}_{4,j}^2} & -\hat{L}_{4,j}/2 \\ \frac{1/2\hat{L}_{4,j}^3}{(\hat{L}_4^2 + \hat{G}_{4,j}^2)^2} & 1 + \frac{1/2\hat{L}_{4,j}^2}{\hat{L}_{4,j}^2 + \hat{G}_{4,j}^2} \end{bmatrix}. \end{aligned}$$

where $L_3 = \tilde{L}_3 + O(\mu) = \hat{L}_3 + O(\mu) = \bar{L}_3 + O(\mu)$, $G_3 = \tilde{G}_3 + O(\mu)$, $\bar{G}_3 = \hat{G}_3 + O(\mu)$, and the notation \simeq means up to $O(\mu)$ error.

(b.4) Finally, the derivative of the renormalization map is

$$d\mathcal{R} = \text{diag} \left\{ \sqrt{\lambda}, 1, -\sqrt{\lambda}, -1; \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{Rot}(\beta), \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\text{Rot}(\beta)}{\sqrt{\lambda}}; -\sqrt{\lambda}, -1 \right\}.$$

Remark 5.1. In the proposition, each of the five matrices is a product of three matrices. For the matrices (I), (III), (V), we use the formula for the derivative of the Poincaré map (see equation (8.1) in Section 8). The matrices M, N_1, N_5 are solutions of the variational equations and the two remaining matrices are boundary contributions coming from the fact that different orbits take different time to travel between two consecutive sections. For (II), we first convert from Delaunay variables to Cartesian variables in the right, then we use $L \cdot R^{-1}$ to convert $(x_3, v_3; x_1, v_1; x_4, v_4)^R \rightarrow (x_3, v_3; x_1, v_1; x_4, v_4)^L$, and finally we convert from Cartesian in the left to Delaunay variables. The matrix (IV) is similar but in the opposite direction.

Remark 5.2. The asymptotics of $O(1)$ blocks in (5.2) is the same as in [DX]. The estimates for N_1, M, N_5 obtained in Section 7 by brute force calculations take a lot of effort. However, only the $O(1)$ blocks contribute to the two leading terms in $d\mathbb{G}$ in Lemma 3.2 (actually, only to the χ^2 part). Our estimates for the remaining entries except the $O(1)$ blocks are only needed to show that their contributions to $d\mathbb{G}$ are controlled by $O(\mu\chi)$. Our estimates in N_1, M, N_5 are actually more than enough to serve this purpose. However, we do not know how bad the estimates of N_1, M, N_5 are allowed to be in order to arrive at the conclusion of Lemma 3.2. See Remark C.1.

The plan of the proof of Proposition 5.1 is as follows. In Section 6 and 7, we write down the equations of motion, the variational equations and estimate their solutions. This gives us the matrices M, N_1 and N_5 in Proposition 5.1. In Section 8, we study the boundary contribution to the derivative the Poincaré map. We get all the u 's and l 's with various sub- and super-scripts. Together with M, N_1, N_5 , the estimates of the boundary contributions complete the estimates of (I), (III), (V). In Section 9, we study the transformation of coordinates from the left to the right and that from the right to the left. This gives us the matrices (II), (IV) stated in Proposition 5.1. The derivative of the renormalization map follows immediately from its definition in Definition 2.4.

We now compute the matrices $R \cdot L^{-1}$ and $L \cdot R^{-1}$ based on Definitions 2.1 and 4.1.

Proof of (5.1). To get $R \cdot L^{-1} = \frac{\partial(x_3, v_3; x_1, v_1; x_4, v_4)^R}{\partial(x_3, v_3; x_1, v_1; x_4, v_4)^L}$, we first use (4.2) to compute the matrix $L^{-1} := \frac{\partial(q_3, p_3; q_1, p_1; q_4, p_4)}{\partial(x_3, v_3; x_1, v_1; x_4, v_4)^L}$, then we use (2.3) to compute $R := \frac{\partial(x_3, v_3; x_1, v_1; x_4, v_4)^R}{\partial(q_3, p_3; q_1, p_1; q_4, p_4)}$. The composition of the two gives us $R \cdot L^{-1}$. Similarly we get $L \cdot R^{-1} = \frac{\partial(x_3, v_3; x_1, v_1; x_4, v_4)^L}{\partial(x_3, v_3; x_1, v_1; x_4, v_4)^R} = (R \cdot L^{-1})^{-1}$. \square

6. EQUATIONS OF MOTION, \mathcal{C}^0 CONTROL OF THE GLOBAL MAP

6.1. The Hamiltonian equations. We solve for L_4 using energy conservation. Since the total energy of the system is zero we get

$$\frac{1}{L_4^2} = \frac{m_3 k_3^2}{m_4 k_4^2 L_3^2} \left(1 - \frac{L_3^2}{m_3 k_3^2} \left(\frac{1}{2m_1} v_1^2 - \frac{k_1}{|x_1|} \right) - \frac{2UL_3^2}{m_3 k_3^2} \right),$$

hence

$$(6.1) \quad L_4 = L_3 \frac{m_4^{1/2} k_4}{m_3^{1/2} k_3} \left(1 + \frac{L_3^2}{2m_3 k_3^2} \left(\frac{1}{2m_1} v_1^2 - \frac{k_1}{|x_1|} \right) + \frac{UL_3^2}{m_3 k_3^2} + h.o.t. \right).$$

We treat ℓ_4 as the new time. So we divide the Hamiltonian equations by the equation $\frac{d\ell_4}{dt} = -\frac{m_4 k_4^2}{L_4^3} + \frac{\partial U}{\partial L_4}$, whose reciprocal is $\frac{dt}{d\ell_4} = -\frac{L_4^3}{m_4 k_4^2} \left(1 + \frac{L_4^3}{m_4 k_4^2} \frac{\partial U}{\partial L_4} + O(U^2) \right)$.

Eliminating L_4 using (6.1) we get

$$(6.2) \quad \begin{aligned} \frac{dt}{d\ell_4} &= -\frac{\left(\frac{m_4^{1/2} k_4}{m_3^{1/2} k_3} \right)^3}{m_4 k_4^2} L_3^3 \left(1 + \frac{L_3^2}{2m_3 k_3^2} \left(\frac{1}{2m_1} v_1^2 - \frac{k_1}{|x_1|} \right) + \frac{3UL_3^2}{m_3 k_3^2} \right) - \frac{\left(\frac{m_4^{1/2} k_4}{m_3^{1/2} k_3} \right)^6}{m_4^2 k_4^4} L_3^6 \frac{\partial U}{\partial L_4} \\ &+ h.o.t. = -L_3^3 \left(1 + L_3^2 \left(\frac{1}{2m_1} v_1^2 - \frac{k_1}{|x_1|} \right) + 3UL_3^2 \right) - L_3^6 \frac{\partial U}{\partial L_4} + h.o.t., \end{aligned}$$

where in the last equality, we use the fact that $k_{3,4}, m_{3,4} = 1 + O(\mu)$ and we absorb the $O(\mu)$ terms into h.o.t.

Now we write the equations of motion as follows:

$$(6.3) \quad \left\{ \begin{array}{l} \frac{dL_3}{d\ell_4} = -\frac{dt}{d\ell_4} \frac{\partial U}{\partial L_3}, \\ \frac{dG_3}{d\ell_4} = -\frac{dt}{d\ell_4} \frac{\partial U}{\partial G_3}, \\ \frac{d\ell_4}{dx_1} = \frac{dt}{d\ell_4} \frac{v_1}{m_1}, \\ \frac{d\ell_4}{dG_4} = -\frac{dt}{d\ell_4} \frac{\partial U}{\partial G_4}, \\ \frac{d\ell_4}{d\ell_4} = -\frac{dt}{d\ell_4} \frac{\partial U}{\partial g_4}, \end{array} \right. \quad \left\{ \begin{array}{l} \frac{d\ell_3}{d\ell_4} = \frac{dt}{d\ell_4} \left(\frac{m_3 k_3^2}{L_3^3} + \frac{\partial U}{\partial L_3} \right), \\ \frac{dg_3}{d\ell_4} = \frac{dt}{d\ell_4} \left(\frac{\partial U}{\partial G_3} \right), \\ \frac{dv_1}{d\ell_4} = -\frac{dt}{d\ell_4} \left(\frac{k_1 x_1}{|x_1|^3} + \frac{\partial U}{\partial x_1} \right), \\ \frac{d\ell_4}{d\ell_4} = \frac{dt}{d\ell_4} \left(\frac{\partial U}{\partial G_4} \right). \end{array} \right.$$

Notation 6.1. We denote the RHS of (6.3) by $\mathcal{F} = (\mathcal{F}_3; \mathcal{F}_1; \mathcal{F}_4)$. Thus (6.3) takes form $\frac{d}{d\ell_4} \mathcal{V}_i = \mathcal{F}_i$, $i = 3, 1, 4$.

6.2. Estimates of the potential U . Here we analyze the power series expansion of U to exhibit certain cancelations. We use the same procedure as in Section 4 based on (4.8). We apply this procedure to all terms in U so that if a term contains x_1 we let a in (4.8) be x_1 and b be the sum of the remaining terms. If a term does not contain x_1 (which is only possible in the right case) we take $a = x_4$.

Lemma 6.1. Suppose that

$$(6.4) \quad |x_1| \geq \chi, \quad |x_4| = O(\chi), \quad |x_3| \leq 2.$$

Then the monomials in our power series satisfy the following estimates.

(1) *Terms containing only x_4, x_1 : these terms have the form*

$$\frac{1}{|x_1|} \frac{\langle x_1, x_4 \rangle^m |x_4|^{2n}}{|x_1|^{2(m+n)}} = O\left(\frac{1}{\chi}\right), \quad m + 2n \geq 2.$$

(2) *Terms containing only x_3, x_1 : these terms have the form*

$$\frac{1}{|x_1|} \frac{\langle x_1, x_3 \rangle^m |x_3|^{2n}}{|x_1|^{2(m+n)}} = O\left(\frac{1}{\chi^{2n+m+1}}\right), \quad \text{with } m + 2n \geq 2.$$

(3) *Terms containing x_1, x_3 and x_4 : these terms must be of order $\frac{1}{\chi^3}$. The power of x_3 is at least 2. When x_3 and x_4 show up simultaneously, there must be an extra factor μ . (One typical term is $\frac{\mu \langle x_3, x_4 \rangle |x_3|}{|x_1|^4}$. The explicit formula for the coefficients is cumbersome and will not be used in the paper).*

(4) *Terms containing only x_3, x_4 have the form $\frac{\mu}{|x_4|} \frac{\langle x_3, x_4 \rangle^m |x_3|^{2n}}{|x_4|^{2(m+n)}}$, $m + 2n \geq 2$. These terms only show up in the right case.*

Proof. The cases (1),(2),(4) follow directly from (4.8) (to get (4) we note that in all U^R terms which do not involve x_1 have μ either in front of x_3 or in the numerator). We have chosen k_1, k_3, k_4 to kill the terms in U with $m = 0, n = 0$. In the Taylor expansions of U , the leading terms are those with $m = 2, n = 0$ and $m = 0, n = 1$ in the above cases (1), (2), (4).

Thus only the case (3) is nontrivial. We claim that in case (3), when both x_1, x_3 , and x_4 are present, the power of x_3 must be greater than 1. To this end we show that the Taylor coefficients of $\langle x_3, x_4 \rangle |x_4|^m$, $m \geq 0$ in the potential are zero. That is

$$\frac{\partial^2}{\partial x_3 \partial x_4} \Big|_{x_3=0} \left[\frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} + \frac{1}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 - \frac{1}{1+\mu} x_3 \right|} \right] = 0.$$

Indeed

$$\frac{\partial^2}{\partial x_3 \partial x_4} \Big|_{x_3=0} \frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} = \frac{\frac{\mu}{(1+2\mu)(1+\mu)}}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 \right|} \left(-\text{Id} + 3 \frac{(x_1 + \frac{\mu}{1+2\mu} x_4)^{\otimes 2}}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 \right|^2} \right)$$

This is exactly $-\frac{\partial^2}{\partial x_3 \partial x_4} \Big|_{x_3=0} \frac{1}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 - \frac{1}{1+\mu} x_3 \right|}$. So the monomials of the form $\langle x_3, x_4 \rangle |x_4|^m$, $m \geq 0$ get canceled. Accordingly in case (3) the power of $|x_3|$ is greater than 1, which implies that the power of $|x_1|$ in the denominator must be at least 3. This gives the $O(1/\chi^3)$ estimate. An extra factor of μ is obtained in the same way as in case (4). \square

6.3. Estimates of the Hamiltonian equations.

6.3.1. Estimates of the positions. The next step in our analysis is an important *a priori* bound. It will be proven in Section 6.4 after we obtain several preliminary estimates. We first introduce a trapezoid to which x_4, x_1 are confined.

Definition 6.2. *If the initial total angular momentum is $G_0 \neq 0$, we let $\mathcal{T}_{G_0, \hat{C}}$ be the trapezoid enclosed by two vertical lines $x_{\parallel} = 0$, $x_{\parallel} = -2\chi$, and two tilting lines,*

$$(6.5) \quad x_{\perp} = \mp \frac{\hat{C}\mu\mathcal{G}}{\chi} x_{\parallel} \pm \hat{C}$$

where \mathcal{G} is defined by (2.7).

In the case of $G_0 = 0$ we let $\mathcal{T}_{0, \hat{C}}$ be the rectangle enclosed by two vertical lines $x_{\parallel} = 0$, $x_{\parallel} = -2\chi$, and two horizontal lines, $x_{\perp} = \pm \hat{C}$.

The next lemma shows that all of the (x_i, v_i) , $i = 3, 1, 4$ behaves like a Kepler motion ($i = 3, 4$) or free motion ($i = 1$) for global map, as if there is no interactions. This fact is intuitively clear, however, has a lengthy proof due to the fact that we have 10 variables to control and have cut the orbit into several pieces.

Lemma 6.2. *Consider an orbit defined on $[0, T]$ with $T = O(\chi)$ such that at the initial and final moments $x_{4,\parallel}^R = -2$ and $x_{4,\perp}^R < -2$ and the orbit hits the section $\{x_{4,\parallel}^R = -\chi/2\}$ at time $\bar{\tau}$ with x_4 moving from right to left, and hits the section $\{x_{4,\parallel}^L = \chi/2\}$ at time $\hat{\tau}$ with x_4 moving from left to right.*

Assume that

- (i) *within the time interval $(0, T)$ we have*

$$(6.6) \quad |x_3| < 2 - \delta, \quad x_4^R \in \mathcal{T}_{G_0, \hat{C}}, \quad -x_4^L \in \mathcal{T}_{G_0, \hat{C}}$$

for some constants $\delta, \hat{C} > 0$ and $\chi > \chi_0$,

- (ii) *at time 0 the initial condition for x_1, v_1 satisfies (2.8).*
 (iii) *and at time 0 the initial conditions for the third and fourth bodies satisfy*

$$(6.7) \quad |E_3(0) + 1/2| \leq C\delta, \quad \frac{1}{C} \leq G_3(0) \leq C, \quad |G_4(0)| \leq C.$$

Then

- (a) *we have*

$$(6.8) \quad \left| \frac{\partial x_3}{\partial \mathcal{V}_3} \right| < C', \quad |E_3(t) - E_3(0)|, \quad |G_3(t) - G_3(0)|, \quad |g_3(t) - g_3(0)| \leq C'\mu, \quad |G_4(t)| \leq C',$$

for $\forall t \in [0, T]$ and some constant C' independent of μ and χ .

- (b) *we have the estimate for the position x_4 , when x_4 is moving to the right of the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$,*

$$(6.9) \quad |x_4^R| \begin{cases} \geq 2, & |\ell_4^*| \leq |\ell_4| \leq C \\ \in \left[\frac{1}{2}, 2 \right] |\ell_4|, & |\ell_4| \geq C, \end{cases}$$

where ℓ_4^ is the value of ℓ_4 restricted on $\{x_{4,\parallel}^R = -2\}$ and C is a constant independent of χ or μ .*

When x_4 is moving to the left of the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$, we have

$$(6.10) \quad |x_4^L| \leq 2|\ell_4| + C.$$

(c) We have in both the left and the right cases

$$(6.11) \quad v_1 = O\left(1, \frac{\mathcal{G}}{\chi}\right), \quad x_1 = ((1 + O(\mu))x_{1,\parallel}(0), O(\mu)\mathcal{G}).$$

Remark 6.1. Part (c) will be used to show that the rotation angle β in Definition 2.4 of the renormalization map is $O(\mu/\chi)$ in the case of zero total angular momentum and is $O(\mu\mathcal{G}/\chi)$ in the case of nonzero angular momentum.

6.3.2. Estimate of the derivatives of the potential.

Lemma 6.3. Define $u(\ell_4) = \frac{1}{\chi^3} + \frac{\mu}{|\ell_4|^3 + 1}$.

(a) If we assume (6.4), then we have the following estimates for the first order derivatives

$$\frac{\partial U^R}{\partial x_3} \lesssim u(\ell_4), \quad \frac{\partial U^R}{\partial x_4} \lesssim \frac{1}{\chi^2} + \frac{\mu}{\ell_4^4 + 1}, \quad \frac{\partial U^{R,L}}{\partial x_1} \lesssim \frac{1}{\chi^2}, \quad \frac{\partial U^L}{\partial x_3} \lesssim \frac{1}{\chi^3}, \quad \frac{\partial U^L}{\partial x_4} \lesssim \frac{1}{\chi^2}.$$

(b) If we assume furthermore that (6.6) holds and in addition $x_1 \in \mathcal{T}_{G_0, \hat{C}}$, then the second order derivatives satisfy the following estimates

$$\begin{aligned} \frac{\partial^2 U^R}{\partial x_3^2} &\lesssim u(\ell_4), \quad \frac{\partial^2 U^R}{\partial x_3 \partial x_4} \lesssim \frac{\mu}{\ell_4^4 + 1}, \quad \frac{\partial^2 U^R}{\partial x_4^2} \lesssim \frac{1}{\chi^3} + \frac{\mu}{|\ell_4|^5 + 1}, \quad \frac{\partial^2 U^{R,L}}{\partial x_3 \partial x_1} \lesssim \frac{1}{\chi^4}, \\ \frac{\partial^2 U^L}{\partial x_3^2} &\lesssim \frac{1}{\chi^3}, \quad \frac{\partial^2 U^L}{\partial x_3 \partial x_4} \lesssim \frac{\mu}{\chi^4}, \quad \frac{\partial^2 U^{R,L}}{\partial x_1^2}, \quad \frac{\partial^2 U^{R,L}}{\partial x_4 \partial x_1}, \quad \frac{\partial^2 U^L}{\partial x_4^2} \lesssim \frac{\text{Id}_2}{\chi^3} + \frac{(\chi, \mu\mathcal{G})^{\otimes 2}}{\chi^5}. \end{aligned}$$

Proof. We use Lemma 6.1. Notice that when we take $\frac{\partial}{\partial x_1}$ (resp. $\frac{\partial}{\partial x_4}$) derivatives the estimates of Lemma 6.1 get multiplied by $\frac{1}{\chi}$ (resp. $\frac{1}{|\ell_4| + 1}$ in the right case and $\frac{1}{\chi}$ in the left case). The estimate remains unchanged if we take $\frac{\partial}{\partial x_3}$ derivatives.

For instance, if we want to estimate $\frac{\partial U^R}{\partial x_3}$, then according to Lemma 6.1 we have contributions from case (2) which are of order $O\left(\frac{1}{\chi^3}\right)$, the contributions of case (3) which are of order $O\left(\frac{\mu}{\chi^3}\right)$ and the contributions of case (4) which are of order $O\left(\frac{\mu}{|\ell_4|^3 + 1}\right)$. If we take $\frac{\partial}{\partial x_3}$ derivative, the estimates remain the same.

We get all the estimates of the lemma using the same procedure. \square

6.3.3. Estimate of the Hamiltonian equation. The next lemma estimates the RHS of (6.3).

Lemma 6.4. Suppose that we have for all the time (6.6), (6.9) and in addition

$$(6.12) \quad \frac{1}{C} \leq |L_3|, |G_3| \leq C, \quad |G_4| \leq C, \quad |x_{1,\parallel}| \geq \chi, \quad |x_{1,\perp}| \leq C\mathcal{G}, \quad |v_{1,\parallel}| \leq C, \quad |v_{1,\perp}| \leq C\mathcal{G}/\chi,$$

then

- (a) we have $\left| \frac{\partial x_3}{\partial \mathcal{V}_3} \right| < C'$ for some constant C' independent of μ and χ .
 (b) In the right case, we have

$$\mathcal{F}^R = (0, 1, 0_{1 \times 8}) + O \left(u(\ell_4), \mu, (u(\ell_4))_{1 \times 2}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\mu \chi^3}; \left(\frac{\mathcal{G}}{\chi^2} + \frac{\mu}{|\ell_4|^3 + 1} \right)_{1 \times 2} \right).$$

- (c) In the left case, we have

$$\mathcal{F}^L = (0, 1, 0_{1 \times 8}) + O \left(\frac{1}{\chi^3}, \mu, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\mu \chi^3}; \frac{\mathcal{G}}{\chi^2}, \frac{\mathcal{G}}{\chi^2} \right).$$

Proof. (a) follows from (A.2) since under the assumptions of the lemma L_3, G_3 are uniformly bounded and x_3 and v_3 depend periodically on g_3 and ℓ_3 .

To prove (b) and (c) we note that (6.1), (6.2) imply that $\frac{dt}{d\ell_4} = O(1)$. To estimate \mathcal{V}_3 part notice that for the Kepler problem only ℓ_3 component is non zero. For other components the estimates remain the same as the estimates for $\frac{\partial U}{\partial x_3}$ due to already proven part (a).

A similar argument applies to \mathcal{V}_1 part under our assumptions on x_1, v_1 . Namely in view of Lemma 6.1 the main contribution to \mathcal{F}_3 comes from the Kepler part.

To get the bound for \mathcal{V}_4 part we use the estimates $\frac{\partial x_4}{\partial g_4} \cdot x_4 = 0$ and $\frac{\partial x_4}{\partial G_4} \cdot x_4 = O(\ell_4)$ from part (c) of Lemma A.2 in Appendix A. For example in the right case in (4.3), we consider derivative of the term

$$\frac{\partial}{\partial G_4} \frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} = \frac{\left(x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right)}{(1+2\mu) \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|^3} \cdot \frac{\partial x_4}{\partial G_4}$$

We claim that the above expression is $O(\mathcal{G}/\chi^2)$, which is covered by the $O(\mathcal{G}/\chi^2)$ part in our bound for \mathcal{F}_4 . Indeed, the denominator is of order χ^3 . The main contributions to the numerator come from $\left\langle x_4, \frac{\partial x_4}{\partial G_4} \right\rangle$ which is $O(\ell_4)$ due to part (c) of Lemma A.2 and (6.9), and from $\left\langle x_1, \frac{\partial x_4}{\partial G_4} \right\rangle$. To estimate the later product we write

$$x_1 = \frac{|x_1|}{|x_4|} \cos \alpha x_4 + |x_1| \sin \alpha \mathbf{e}$$

where $\alpha = \angle(x_4, x_1)$ and \mathbf{e} is the unit vector perpendicular to x_4 . In the case of zero total angular momentum, we note that the assumptions (6.6) and (6.9) imply $\alpha = O(1/\ell_4)$. This gives

$$\left\langle x_1, \frac{\partial x_4}{\partial G_4} \right\rangle = O \left(\frac{|x_1|}{|x_4|} \right) \left\langle x_4, \frac{\partial x_4}{\partial G_4} \right\rangle + |x_1| O(\alpha) O \left(\left| \frac{\partial x_4}{\partial G_4} \right| \right) = O(\chi)$$

where the last estimate comes from Lemma A.2(c). In the case of nonzero angular momentum, the above estimates have to be modified as follows: $\angle(x_4, x_1) =$

$O(1/\ell_4 + \mathcal{G}/\chi)$ and $\left\langle x_1, \frac{\partial x_4}{\partial G_4} \right\rangle = O(\chi\mathcal{G})$. The other derivatives are estimated similarly and result in the estimates of the lemma. In particular, the $O\left(\frac{\mu}{|\ell_4|^3 + 1}\right)$ part in our bound for \mathcal{F}_4 comes from differentiating the term in U^R which do not contain x_1 . This bound is obtained by multiplying the $O\left(\frac{\mu}{|\ell_4|^4 + 1}\right)$ term in the estimate of $\frac{\partial U^R}{\partial x_4}$ in part (a) of Lemma 6.3 by $O(\ell_4)$ bound for $\frac{\partial x_4}{\partial G_4}$ coming from Lemma A.2(c). \square

Sometimes, we do not have the boundedness of G_4 a priori. However, we can still get the same estimates as in Lemma 6.2 for $\frac{d}{d\ell_4}(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4)$ (that is we are missing only the estimate for $\frac{d}{d\ell_4}g_4$) without making any assumptions on G_4 .

Corollary 6.1. (a) *In the left case, if we assume (6.6) plus*

$$(6.13) \quad \frac{1}{C} \leq |L_3|, |G_3| \leq C, \quad |x_{1,\parallel}| \geq \chi, \quad |x_{1,\perp}| \leq C\mathcal{G}, \quad |v_{1,\parallel}| \leq C, \quad |v_{1,\perp}| \leq C\mathcal{G}/\chi$$

then we have the estimates

$$\begin{aligned} & \frac{d}{d\ell_4}(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4)^L \\ &= (0, 1, 0_{1 \times 7}) + O\left(\frac{1}{\chi^3}, \mu, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu\mathcal{G}}{\chi}, \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\mu\chi^3}; \frac{\mathcal{G}}{\chi^2}\right). \end{aligned}$$

(b) *In the right case, assume (6.6), (6.13) and in addition $|x_4| \geq C|\ell_4| + 1$, then we get the estimates*

$$\begin{aligned} & \frac{d}{d\ell_4}(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4)^R \\ &= (0, 1, 0_{1 \times 7}) + O\left(u(\ell_4), \mu, u(\ell_4), u(\ell_4); \mu, \frac{\mu\mathcal{G}}{\chi}, \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\mu\chi^3}; \frac{\mathcal{G}}{\chi^2} + \frac{\mu}{|\ell_4|^3 + 1}\right). \end{aligned}$$

Proof. We only need to do the $\frac{dG_4}{d\ell_4}$ estimate. The others are the same as before and do not use the G_4 bound. We have $\frac{dG_4}{d\ell_4} = -\frac{dt}{d\ell_4} \frac{\partial U}{\partial x_4} \frac{\partial x_4}{\partial g_4}$. Notice $\frac{\partial x_4}{\partial g_4}$ is a $\pi/2$ rotation of the vector x_4 and we have $\angle(x_4, x_1) = O(1/\ell_4 + \mathcal{G}/\chi)$ from assumption (6.6). Hence we have $\left\langle x_1, \frac{\partial x_4}{\partial g_4} \right\rangle = O(\chi\mathcal{G})$ in the same way as we did in the proof of Lemma 6.4. We then use Lemma 6.1 (mainly items (1) and (4)) to get $\frac{dG_4}{d\ell_4}$ estimate. \square

6.4. Proof of Lemma 6.2.

Proof. We prove the estimates of the lemma on time intervals $[0, \bar{\tau}]$, $[\bar{\tau}, \hat{\tau}]$ and $[\hat{\tau}, T]$ separately.

Step 1, the interval $[0, \bar{\tau}]$.

In this step, all the variables should carry a superscript R , but we omit it for simplicity. Let τ be the maximal time interval such that

$$(6.14) \quad \begin{aligned} \frac{G_3(\ell_4)}{G_3(\ell_4^*)}, \frac{L_3(\ell_4)}{L_3(\ell_4^*)}, \frac{v_{1,\parallel}(\ell_4)}{v_{1,\parallel}(\ell_4^*)} &\in \left[\frac{3}{4}, \frac{4}{3} \right], \quad |G_4(\ell_4)| \leq 2|G_4(\ell_4^*)| + 1, \\ |x_{1,\perp}(\ell_4)| &\leq \mathcal{G}, \quad v_{1,\perp}(\ell_4) \leq \frac{2C\mathcal{G}}{\chi}, \end{aligned}$$

where ℓ_4^* is the value ℓ_4 restricted on $\{x_4 = -2\}$.

We always have $|x_4| \geq 2$ since x_4 is to the left of the section $\{x_4 = -2\}$. So we get $L_4 = L_3 + O(\mu)$ using (6.1). Then using formula (A.5) and $e_4 = \sqrt{1 + \frac{G_4^2}{L_4^2}}$, we find

$$(6.15) \quad \begin{aligned} |x_4| &= L_4 \sqrt{L_4^2 (\cosh u - e_4)^2 + G_4^2 \sinh^2 u} \\ &= L_4 \sqrt{L_4^2 (\cosh^2 u - 2e_4 \cosh u + e_4^2) + (L_4^2 e_4^2 - L_4^2) \sinh^2 u} \\ &= L_4^2 \sqrt{1 - 2e_4 \cosh u + e_4^2 + e_4^2 \sinh^2 u} \\ &= L_4^2 \sqrt{1 - 2e_4 \cosh u + e_4^2 + e_4^2 (\cosh^2 u - 1)} \\ &= L_4^2 \sqrt{(1 - e_4 \cosh u)^2} = L_4^2 (e_4 \cosh u - 1) \end{aligned}$$

We always have $e_4 \geq 1$, so we get $|\ell - u| \geq |\sinh u| \geq \frac{e^{|u|} - 1}{2}$ from (A.4), so that $u = o(\ell)$ as $|\ell| \rightarrow \infty$. Continuing (6.15), we have

$$e_4 \cosh u \geq e_4 |\sinh u| = |\ell - u| = (1 + o(1))|\ell|.$$

So we obtain

$$(6.16) \quad |x_4| \geq L_4^2 (1 + o(1)) |\ell_4|, \quad \text{as } |\ell_4| \rightarrow \infty.$$

We have that $|G_4(\ell_4^*)|$ is bounded since both x_4, v_4 are bounded at the initial moment, where the boundedness of v_4 comes from the fact that the initial energies of x_1, x_3 are bounded. Combined with (6.14) and the assumption that the initial energy of x_3 is $1/2 + O(\delta)$, the estimate (6.16) proves estimate (6.9) on the interval $[0, \min(\tau, \bar{\tau})]$.

Thus on the time interval $[0, \min(\tau, \bar{\tau})]$, the assumptions of Lemma 6.4 are satisfied and hence $\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4} \lesssim u(\ell_4)$.

Next, consider the (x_1, v_1) component. Under the assumption (6.14), we get the estimate $\frac{dv_1}{d\ell_4} = O\left(\frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\mu\chi^3}\right)$ using Lemma 6.4. Over time $\ell_4 = O(\chi)$, the total oscillation of v_1 is $O\left(\frac{1}{\mu\chi}, \frac{\mathcal{G}}{\mu\chi^2}\right)$, which is much smaller than the initial value $v_1 = O(1, \mathcal{G}/\chi)$ according to (2.9). We get $v_1(\ell_4) = v_1(\ell_4^*) + O\left(\frac{1}{\mu\chi}, \frac{\mathcal{G}}{\mu\chi^2}\right)$. We then use (6.3) to get the estimate $\frac{dx_1}{d\ell_4} = O(\mu)(v_{1,\parallel}, v_{1,\perp})$ (noticing that $m_1 \sim 1/\mu$ in (6.3)). Next we integrate over time $\ell_4 = O(\chi)$ to get the estimates

$$x_1(\ell_4) = x_1(\ell_4^*) + O(\mu)(\chi, v_{1,\perp}\chi) = ((x_1(\ell_4^*) + O(\mu\chi), O(\mu\mathcal{G})).$$

Therefore $\bar{\tau} < \tau$ so (6.14) holds on the interval $[0, \bar{\tau}]$. Hence we can integrate ℓ_4 over interval of order χ and get that the oscillation of L_3, G_4, G_3 is $O(\mu)$ on that interval. So the assumption (6.14) can be dropped proving the estimates of the lemma before time $\bar{\tau}$. To summarize, we get that during the interval $[0, \bar{\tau}]$,

$$(6.17) \quad \begin{aligned} x_1^R &= ((1 + O(\mu))\chi, O(\mu\mathcal{G})), \quad v_{1,\perp}^R = v_{1,\perp}^R(0) + O\left(\frac{\mu\mathcal{G}}{\chi}\right), \quad v_{1,\parallel}^R < 0, \\ |L_3 - L_3(0)|, |G_3 - G_3(0)|, |g_3 - g_3(0)|, |G_4^R - G_4^R(0)| &= O(\mu). \end{aligned}$$

The superscript R or L is not needed for L_3, G_3, g_3 .

Step 2, the interval $[\bar{\tau}, \hat{\tau}]$.

When the orbit enters the left of the section $\{x_{4,\parallel}^R = -\chi/2\}$, we have $(L_3, G_3, g_3)(\bar{\tau})$ satisfy (6.17). Since the oscillation of G_4^R is $O(\mu)$ in (6.17), we have estimates of the angular momentum

$$G_4^R = v_4^R \times x_4^R = v_{4,\parallel}^R x_{4,\perp}^R - x_{4,\parallel}^R v_{4,\perp}^R = O(1).$$

This implies

$$(6.18) \quad v_4^R = O(1, \mu\mathcal{G}/\chi)$$

on the section $\{x_{4,\parallel}^R = -\chi/2\}$ since we have $x_4^R = (-\chi/2, O(\mu\mathcal{G}))$ using assumption (6.6) and $|v_{4,\parallel}^R| > c > 0$ due to energy conservation. We also have $v_{4,\parallel}^R < 0$, $v_{1,\parallel}^R < 0$. Next, we use $L \cdot R^{-1}$ in Proposition 5.1 to convert the right variables to the left and get

$$(6.19) \quad x_1^L = \frac{1}{1+\mu}x_1^R + \frac{2\mu}{1+2\mu}x_4^R, \quad v_1^L = \frac{1+\mu}{1+2\mu}v_1^R + v_4^R.$$

So we get the initial conditions for the piece the interval $[\bar{\tau}, \hat{\tau}]$ using (6.19)

$$x_1^L(\bar{\tau}) = x_1^R(0) + O(\mu\chi, \mu\mathcal{G}), \quad v_{1,\perp}^L(\bar{\tau}) = (1 + O(\mu))v_{1,\perp}^R(0) + O\left(\frac{\mu\mathcal{G}}{\chi}\right), \quad v_{1,\parallel}^L < 0$$

restricted on the section $\{x_{4,\parallel}^R = -\chi/2\}$. However, we could not bound G_4^L by a constant now. We make the assumption

$$(6.20) \quad \frac{G_3(\ell_4)}{G_3(\ell_4^*)}, \frac{L_3(\ell_4)}{L_3(\ell_4^*)}, \frac{v_{1,\parallel}(\ell_4)}{v_{1,\parallel}(\ell_4^*)} \in \left[\frac{3}{4}, \frac{4}{3}\right], \quad |x_{1,\perp}(\ell_4)| \leq \mathcal{G}, \quad v_{1,\perp}(\ell_4) \leq \frac{2C\mathcal{G}}{\chi},$$

where ℓ_4^* is the value ℓ_4 restricted on $\{x_4^R = -\chi/2\}$. These assumptions allow us to establish the estimates corresponding to $\frac{d}{d\ell_4}(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4)$ in Corollary 6.1 part (a). We use the same argument as Step 1 to get the estimates on the interval $[\bar{\tau}, \hat{\tau}]$

$$(6.21) \quad \begin{aligned} x_1^L &= ((1 + O(\mu))\chi, O(\mu\mathcal{G})), \quad v_{1,\perp}^L = (1 + O(\mu))v_{1,\perp}^R(0) + O\left(\frac{\mu\mathcal{G}}{\chi}\right), \quad v_{1,\parallel}^L < 0, \\ |L_3 - L_3(\bar{\tau})|, |G_3 - G_3(\bar{\tau})|, |g_3 - g_3(\bar{\tau})| &= O(1/\chi^2), \quad |G_4^L - G_4^L(\bar{\tau})| = O(\mathcal{G}/\chi). \end{aligned}$$

This shows that (6.20) is automatically satisfied on $[\bar{\tau}, \hat{\tau}]$ so we do not need to have it as an additional assumption.

Step 3, bounding G_4^L and v_4^L on the section $\{x_{4,\parallel}^L = \chi/2\}$.

To enter the last piece $[\hat{\tau}, T]$, we need estimates of the initial conditions on x_1^R, v_1^R . We want to apply the same argument as Step 2 with R and L switching roles, so we must have an estimate for v_4^L playing the same role as (6.18). However, we do not have the $O(1)$ estimate on G_4^L on the section $\{x_{4,\parallel}^L = \chi/2\}$ which is needed to get an estimate for v_4^L .

First we establish a uniform upper bound for $|G_4^L|$ using the assumption $-x_4^L \in \mathcal{T}_{G_0, \hat{C}}$ in (6.6). This assumption means that x_4^L is a small perturbation of the Kepler motion. In particular, during the interval when $x_{4,\parallel} \in [-\frac{\chi}{2}, -\frac{\chi}{4}]$ we get $\dot{v}_4 = O(1/\chi^2)$ so v_4 oscillates by $O(1/\chi)$. Accordingly the motion of x_4^L is close to linear and since $-x_4^L \in \mathcal{T}_{G_0, \hat{C}}$ we conclude that the initial velocity of x_4^L is almost horizontal and so $|x_4^L|$ decreases with linear speed. Next let τ^\dagger be the first time when $|x_4^L| = \sqrt{\chi}$. Then for $t \in [\bar{\tau}, \tau^\dagger]$ we have

$$\dot{v}_{4,\perp} = O\left(\frac{x_{4,\perp}}{|x_4|^3}\right) = O\left(\frac{\varepsilon}{\sqrt{\chi}|x_4|^2}\right)$$

so the oscillation of $v_{4,\perp}$ on this interval is $O(\frac{\sqrt{\varepsilon}}{\sqrt{\chi}} \frac{1}{\sqrt{\chi}}) = O(\frac{\varepsilon}{\chi})$. Next, on one hand $x_{4,\perp}(\tau^\dagger) = O(1)$ since $-x_4^L \in \mathcal{T}_{G_0, \hat{C}}$. On the other hand

$$x_{4,\perp}(\tau^\dagger) = x_{4,\perp}^*(\tau^\dagger) + v_{4,\perp}^*(\tau^\dagger - \bar{\tau}) + O\left(\frac{\varepsilon}{\sqrt{\chi}} \ln \chi\right).$$

Hence $v_4(\tau^\dagger - \bar{\tau}) = O(\sqrt{\chi})$ and since $\tau^\dagger - \bar{\tau} \geq c\chi$ we conclude that $v_{4,\perp} = O(\frac{1}{\sqrt{\chi}})$ on $[\bar{\tau}, \tau^\dagger]$. Hence $G_4^L(\tau^\dagger) = O(1)$ and by the last estimate of (6.21), $G_4^L = O(1)$ on $[\bar{\tau}, \hat{\tau}]$.

In fact, we have a stronger estimate $G_4^L = o(1)$ on $[\bar{\tau}, \hat{\tau}]$. Indeed, let $[\tau^\dagger, \tau^\ddagger]$ be the maximal interval where $|x_4^L| \leq \sqrt{\chi}$. On this time interval x_4^L is $O(\frac{1}{\chi^2})$ perturbation of the Kepler motion whose energy is bounded from above and below and whose angular momentum is $O(1)$. So the motion of x_4^L is $O(\frac{1}{\chi})$ close to the hyperbola whose asymptotes make angle $2 \arctan\left(\frac{G_4^L(\tau^\dagger)}{L_4(\tau^\dagger)}\right)$. Since $x_{4,\perp}$ remains $O(1)$ on $[\tau^\dagger, \tau^\ddagger]$ we conclude that actually $G_4^L = O\left(\frac{1}{\sqrt{\chi}}\right)$.

Now we get (6.10) using the last line of (6.15).

Next, we get

$$(6.22) \quad v_4^L = O(1, \mu \mathcal{G}/\chi)$$

using assumption (6.6) and the fact that $G_4^L = O(1)$ in the same way as establishing (6.18) in Step 2. Moreover, we compare (4.5) on the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$. Energy conservation and the estimates (6.21) on L_3^L, v_1^L imply that the oscillation of $|v_4^L|$ and hence that of $|v_{4,\parallel}^L|$ are $O(\mu)$.

Step 4, preparing initial data for the interval $[\hat{\tau}, T]$.

Now we are ready to convert to the right to get x_1^R, v_1^R on the section $\{x_{4,\parallel}^L = \chi/2\}$. We use $R \cdot L^{-1}$ in Proposition 5.1 to get

$$(6.23) \quad x_1^R = \frac{1+\mu}{1+2\mu}x_1^L - \frac{2\mu}{1+2\mu}x_4^L, \quad v_1^R = \frac{1}{1+\mu}v_1^L - v_4^L.$$

The estimate of v_4^L is done in (6.22). The estimates of x_1^L, v_1^L are done in (6.21) and x_4^L is controlled by Assumption (6.6). Plugging our estimates for $x_4^L, v_4^L, x_1^L, v_1^L$ into (6.23), we get that

$$x_1^R(\hat{\tau}) = ((1+O(\mu))\chi, O(\mu\mathcal{G})), \quad v_{1,\perp}^R(\hat{\tau}) = (1+O(\mu))v_{1,\perp}^R(0) + O\left(\frac{\mu\mathcal{G}}{\chi}\right), \quad v_{1,\parallel}^R < 0,$$

as the initial condition for the final piece $[\hat{\tau}, T]$, where the last estimate is obtained applying (6.23) to $v_{1,\parallel}^L < 0$ in (6.21) and $v_{4,\parallel}^L > 0$. The estimate of $|v_{1,\parallel}^R|$ is $O(1)$ since the oscillation of $|v_4|$ is $O(\mu)$ as we saw at the end of Step 3.

Step 5, the interval $[\hat{\tau}, T]$.

Again we do not have an *a priori* bound for G_4^R . We make the assumption (6.20) where ℓ_4^* is the value ℓ_4^R restricted on $\{x_4^L = -\chi/2\}$ and then integrate the estimates for $\frac{d}{d\ell_4}(L_3, G_3, g_3; x_1, v_1; G_4)$ according to Corollary 6.1(b) (Notice that (6.16) implies $|x_4| \geq C|\ell_4| + 1$ as required by Corollary 6.1). We get on the interval $[\hat{\tau}, T]$ that

$$(6.24) \quad \begin{aligned} x_1^R &= ((1+O(\mu))\chi, O(\mu\mathcal{G})), \quad v_{1,\perp}^R = (1+O(\mu))v_{1,\perp}^R(0) + O\left(\frac{\mu\mathcal{G}}{\chi}\right), \quad v_{1,\parallel}^R < 0, \\ |L_3 - L_3(\hat{\tau})|, |G_3 - G_3(\hat{\tau})|, |g_3 - g_3(\hat{\tau})|, |G_4^R - G_4^R(\hat{\tau})| &= O(\mu) \end{aligned}$$

and remove the extra assumption (6.20) as before.

Finally we bound $|G_4^R|$. We only need to establish $|G_4^R| < C$ on the section $\{x_{4,\parallel}^R = -2\}$ since we know the oscillation of G_4^R is $O(\mu)$ according to (6.24). Indeed, we already bound $|L_3|$ from above and below and bound $|v_1^R|$ at the end of Step 4. Hence we get $|v_4^R| = O(1)$ on the section $\{x_{4,\parallel}^R = -2\}$ using energy conservation. Assumption (6.6) implies the boundedness of $|x_4|$ on the section $\{x_{4,\parallel}^R = -2\}$. So we get $G_4^R = O(1)$ on the section $\{x_{4,\parallel}^R = -2\}$.

Step 6, part (a) of the lemma.

To show part (a), we notice $\frac{\partial x_3}{\partial \mathcal{V}_3}$ depends on ℓ_3, g_3 periodically according to equation (A.2). So part (a) follows since we already got bounds for L_3 and G_3 . \square

Since the estimate of $|x_{1,\perp}| = O(\mu\mathcal{G})$ in part (c) of Lemma 6.2 improves the assumption $|x_{1,\perp}| \leq O(\mathcal{G})$ of Lemma 6.4, we get the following corollary of Lemma 6.2 improving Lemma 6.4.

Corollary 6.2. *Suppose that we have (6.6), (6.9) and in addition*

$$(6.25) \quad \frac{1}{C} \leq |L_3|, |G_3| \leq C, \quad |G_4| \leq C, \quad |x_{1,\parallel}| \geq \chi, \quad |x_{1,\perp}| \leq C\mu\mathcal{G}, \quad |v_{1,\parallel}| \leq C, \quad |v_{1,\perp}| \leq C\mathcal{G}/\chi,$$

then

(a) in the right case, we have

$$\mathcal{F}^R = (0, 1, 0_{1 \times 8}) + O\left(u(\ell_4), \mu, u(\ell_4), u(\ell_4); \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; v(\ell_4), v(\ell_4)\right),$$

$$\text{where we define } v(\ell_4) = \frac{\mu \mathcal{G}}{\chi^2} + \frac{\mu}{|\ell_4|^3 + 1},$$

(b) in the left case, we have

$$\mathcal{F}^L = (0, 1, 0_{1 \times 8}) + O\left(\frac{1}{\chi^3}, \mu, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2}\right).$$

Proof. Our assumption here on $x_{1,\perp}$ is stronger than that in Lemma 6.4. We follow the proof of Lemma 6.4 noticing that the only difference in the proof is that we get now $\angle(x_4, x_1) = O(1/\ell_4 + \mu \mathcal{G}/\chi)$ and $\left\langle x_1, \frac{\partial x_4}{\partial G_4} \right\rangle, \left\langle x_1, \frac{\partial x_4}{\partial g_4} \right\rangle = O(\mu \chi \mathcal{G})$, which improves the $O(\chi \mathcal{G})$ estimate that we had before. \square

We use the assumptions of this corollary as the standing assumptions to estimate Hamiltonian equations and variational equations. These assumptions confine x_1, x_4 such that their motions are close to linear motion forming a small angle with the x_{\parallel} axis. These assumptions are satisfied if Lemma 6.2 holds.

6.5. Justification of the assumptions of Lemma 6.2. We demonstrate that the orbits satisfying **AG** satisfy the assumptions of Lemma 6.2. In **AG** we make assumptions on the initial and final values of x_4, v_4 . However, in the assumptions of Lemma 6.2, we require for all time the orbit of x_4 to be bounded in $\mathcal{T}_{G_0, \hat{C}}$.

Lemma 6.5. Fix δ, C . Consider a time interval $[0, T]$ and an orbit satisfying the following conditions

- (i) $x_{4,\parallel}^R(t) < -2$ for $t \in (0, T)$, and $x_{4,\parallel}^R(0) = x_{4,\parallel}^R(T) = -2$.
- (ii) $x_{4,\perp}^R(0), G_4^R(0), x_{4,\perp}^R(T), G_4^R(T) \leq C$.
- (iii) At time 0, $|E_3(0) + 1/2| \leq C\delta$, $\sqrt{2}/2 < e_3(0) < 1$ and x_1, v_1 satisfies (2.9).

Then there exist constants \hat{C}, μ_0 such that for $\mu \leq \mu_0$ we have

$$x_1, x_4^R, -x_4^L \in \mathcal{T}_{G_0, \hat{C}}, \text{ for all } t \in [0, T].$$

Proof. **Step 1, v_4 has nonzero horizontal component.**

Sublemma 6.6. Given $\tilde{\theta}$ there exist μ_0, χ_0 such that under the assumptions of Lemma 6.5 if $\mu \leq \mu_0, \chi \geq \chi_0$ and $|E(t)| \leq C$ for all $t \in [0, T]$ then

$$(6.26) \quad |\pi - \theta_4^+(t)| < \tilde{\theta}$$

for all $t \leq \bar{\tau}$ where θ_4^+ is the slope of the outgoing asymptotata of Q_4 and $\bar{\tau}$ is the first time when $x_4^R = -\chi/2$.

Proof. The idea of the proof is to show that the segment of orbit with $\{D \leq |x_{4,\parallel}^{R,L}| \leq \chi/2\}$ for some large D is approximately linear. For linear motion, the slope can only be $O(\mu \mathcal{G}/\chi)$ since Q_4 would come close to Q_1 and $x_{1,\perp} = O(\mu \mathcal{G})$ as we will show later. We elaborate the idea as follows.

Pick a large D and let τ^* be the first time when $|x_4^R(\tau^*)| = D$. It is enough to consider below the times $t \geq \tau^*$ since θ_4^+ changes little on the time segment $[0, \tau^*]$ because the motion of Q_4 is a small perturbation of the Kepler motion. Next,

$$\theta^+(\tau^*) = \arctan\left(\frac{v_{4,\perp}^R}{v_{4,\parallel}^R}\right)(\tau^*) + o_{D \rightarrow \infty}(1).$$

To fix our idea we suppose that $\arctan\left(\frac{v_{4,\perp}^R}{v_{4,\parallel}^R}\right)(\tau^*) \geq -\frac{\pi}{4}$. (If $\arctan\left(\frac{v_{4,\perp}^R}{v_{4,\parallel}^R}\right)(\tau^*) < -\frac{\pi}{4}$, that is velocity is almost vertical, the argument is similar but simpler.)

Let τ^\dagger be the first time when $|v_4^R(\tau^\dagger) - v_4^R(\tau^*)| > 0.01$. For $t \leq \min(\bar{\tau}, \tau^\dagger)$ we have

$$D + c(t - \tau^*) < |x_4^R(t)| < D + C(t - \tau^*).$$

On the other hand, the Hamiltonian equations give

$$\dot{v}_4^R = -(1 + O(\mu)) \frac{x_4^R + O(\mu x_3)}{|x_4^R|^3} + O\left(\frac{x_4^R}{\chi^3}\right),$$

where x_3 is bounded since we assumed the boundedness of $|E_3|$. Integrating this estimate we get

$$|v_4^R(t) - v_4^R(0)| \leq 1.1 \int_0^t \frac{1}{|D + cs|^2} + O\left(\frac{D + Cs}{\chi^3}\right) ds = \frac{1.1}{cD} + O(t^2/\chi^3).$$

Thus, the oscillation of v_4^R is smaller than $\frac{2}{cD}$ if $t \leq \tau^\dagger$ and $t = O(\chi)$. It follows that $\bar{\tau} = O(\chi)$ and $\tau^\dagger > \bar{\tau}$. Integrating the \dot{x}_1^R, \dot{v}_1^R equations until time $\bar{\tau} = O(\chi)$ we get

$$(6.27) \quad x_1^R = x_1^R(0) + O(\mu)(\chi, \mathcal{G}) \text{ and } v_1^R = v_1^R(0) + (1/(\mu\chi), \mathcal{G}/(\mu\chi^2))$$

Next, we use (6.19) to get

$$(6.28) \quad \left| \arctan \frac{v_{4,\perp}^L}{v_{4,\parallel}^L}(\bar{\tau}) - \arctan \frac{v_{4,\perp}^R}{v_{4,\parallel}^R}(\tau^*) \right| \leq \frac{5}{cD} + O(\mu)$$

and $|x_{4,\perp}^L| \geq \frac{\chi}{2}\tilde{\theta}/2 - \frac{5}{cD} + O(\mu)$. Next, we consider the left piece of orbit. We claim that in order for Q_4 not to escape, we must have $|x_{4,\perp}^L(\tau^{**})| = O(1)$ where τ^{**} is the first time when $\{x_{4,\parallel}^L = D\}$. Indeed if $x_{4,\perp}^L(\tau^{**})$ is large, then an argument similar to the one given above shows that after the time $O(\chi)$ our system splits into three parts. Namely, Q_1 , Q_4 and $Q_2 - Q_3$ pair will be at distance at least $c\chi$ from each other, they will move with approximately constant speed in such a way that thier mutual distances increase linearly (to control the motion of Q_4 we use (2.9) and the fact that if $x_{4,\perp}^L(\tau^{**})$ is large then the energy and momentum exchange between Q_1 and Q_4 is small due to the computations presented above). Now the standard perturbation theory shows that Q_4 escapes proving our claim.

Next, (6.28) shows that $v_4^R(\bar{\tau})$ is almost horizontal and hence $v_4^R(t)$ for $t \in [\tau^*, \bar{\tau}]$ is almost horizontal. \square

Step 2, estimate of x_1^R, v_1^R , and L_3 .

In the proof of the sublemma we already have $v_1^R = O(1, \mathcal{G}/\chi)$, $x_1^R \in \mathcal{T}_{G_0, \hat{C}}$ under the assumption $|E_3| < C$ for the piece of orbit to the right of the section $\{x_{4,\parallel}^R = -\chi/2\}$.

We then show that the oscillation of L_3 is $O(\mu)$ during the interval $(0, T)$ so that we get rid of the assumption $|E_3| < C$ in the sublemma. We consider the time interval $[0, \tau]$ where $\tau := \sup\{t : |L_3(t) - L_3(0)|, |G_3(t) - G_3(0)| < \delta\}$. During the time interval we have $|x_3| < 2 - \delta$ if δ is small enough, since L_3^2 is close to 1 as the semimajor axis and the initial eccentricity is less than 1 as assumed. This avoids the possibility of collisions between Q_3 and Q_4 since $x_4 < -2$. Energy conservation shows $|v_4^R| \geq 1/2$ on $[0, \min\{\tau, \bar{\tau}\}]$ where $\bar{\tau}$ is the time when the orbit hits the section $\{x_{4,\parallel}^R = -\chi/2\}$. Sublemma 6.6 implies that $|v_{4,\parallel}| > c > 0$, therefore $|x_4|$

grows linearly. We get the estimate $\dot{L}_3, \dot{G}_3 = O\left(\frac{\mu}{t^3 + 1} + O(1/\chi^3)\right)$ using Lemma 6.1 (mainly item (2)) on $[0, \min\{\tau, \bar{\tau}\}]$. So we get the oscillation of L_3, G_3 is $O(\mu)$ during the time interval $[0, \bar{\tau}]$ and $\tau \geq \bar{\tau}$. Next, we consider the left piece of orbit. We have $v_1^L = O(1, \mathcal{G}/\chi)$ by integrating the \dot{v}_1 equation (see (6.3)) and using (6.19) to estimate the initial condition on the interval $[\bar{\tau}, \hat{\tau}]$ where $\hat{\tau}$ is the time when the orbit hits the section $\{x_{4,\parallel}^L = \chi/2\}$. On the time interval $[\bar{\tau}, \min\{\hat{\tau}, \tau\}]$, we have $\dot{L}_3, \dot{G}_3 = O(1/\chi^3)$ and $|v_4^L| \geq 1/2$ using energy conservation. We get $O(1/\chi^2)$ oscillation of L_3, G_3 and $\tau \geq \hat{\tau}$. For the last piece of orbit $[\hat{\tau}, T]$, we apply the same argument as on $[0, \bar{\tau}]$ to the time reversed orbit and the estimate $\dot{L}_3, \dot{G}_3 = O\left(\frac{\mu}{(T-t)^3 + 1} + O(1/\chi^3)\right)$ to show that the oscillation of L_3, G_3 is $O(\mu)$ and $\tau \geq T$.

Step 3, the bound on $(x_4, v_4)^{R,L}$ on the middle sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$.

We continue to work on x_4 . We first show the boundedness of $G_4^{R,L}$. We have $\dot{G}_4 = \frac{\partial U}{\partial x_4} \frac{\partial x_4}{\partial g_4} = O(1/\chi)$ directly from Lemma 6.1 (mainly item (1)), the bound on L_3 , and the fact that $\left|\frac{\partial x_4}{\partial g_4}\right| = |x_4|$ without any assumption on G_4 . This implies the oscillation of G_4 is $O(1)$ over time $O(\chi)$. Similarly to Sublemma 6.6 we see that in the left case the slope of asymptotes of x_4^L are bounded by $\tilde{\theta}$ since otherwise Q_4 will escape after turning around Q_1 . This implies the eccentricity $e = \sqrt{1 + (G/L)^2}$ is close to 1 therefore $G_4^L = O(1)$ when Q_4 comes close to Q_1 . We also assumed that $G_4^R = O(1)$ on the section $\{x_{4,\parallel}^R = -2\}$. We get $G_4^L, G_4^R = O(1)$ for all the time when they are defined, in particular, when evaluated on the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$.

We use (6.19) to get the relation for angular momentums

$$\begin{aligned}
 G_4^L &= v_4^L \times x_4^L = \left(\frac{1}{1+\mu} v_4^R - \frac{2\mu}{1+2\mu} v_1^R \right) \times \left(\frac{x_4^R(1+\mu)}{1+2\mu} - x_1^R \right) \\
 &= \frac{G_4^R}{1+2\mu} - \frac{1}{1+\mu} v_4^R \times x_1^R - \frac{2\mu(1+\mu)}{(1+2\mu)^2} v_1^R \times x_4^R + \frac{2\mu}{1+2\mu} v_1^R \times x_1^R.
 \end{aligned}
 \tag{6.29}$$

Using the estimates on v_1^R, x_1^R in Step 2, we get on the section $\{x_{4,\parallel}^R = -\chi/2\}$

$$O(1) = G_4^L - \frac{G_4^R}{1+2\mu} = (1+O(\mu))[v_{4,\perp}^R\chi + O(\mu\mathcal{G})] + O(\mu)[\mathcal{G} + x_{4,\perp}^R] + O(\mu)[O(\mathcal{G})].$$

This implies

$$(6.30) \quad v_{4,\perp}^R\chi = O(\mu\mathcal{G}) + O(\mu)x_{4,\perp}^R.$$

Next, we have

$$(6.31) \quad O(1) = G_4^R = v_{4,\perp}^R x_{4,\parallel}^R - x_{4,\perp}^R v_{4,\parallel}^R = v_{4,\perp}^R\chi/2 - x_{4,\perp}^R v_{4,\parallel}^R.$$

Substituting (6.30) into (6.31) and using the lower bound on $v_{4,\parallel}^R$ we get $x_{4,\perp}^R = O(\mu\mathcal{G})$. We next substitute the $x_{4,\perp}^R$ estimate back into (6.30) to get $v_{4,\perp}^R = O(\mu\mathcal{G}/\chi)$. We then obtain $x_{4,\perp}^L = O(\mu\mathcal{G})$ and $v_{4,\perp}^L = O(\mu\mathcal{G}/\chi)$ using (6.23). Remember that these estimates are only established so far on the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$.

Step 4, bounding the right piece of the orbit x_4^R .

We next bound the orbit going from the section $\{x_{4,\parallel}^R = -D\}$ to the section $\{x_{4,\parallel}^R = -\chi/2\}$ for some large constant D independent of χ, \mathcal{G}, μ . We have initial condition $|x_{4,\perp}(t_0)| \leq \hat{C}$ on the section $\{x_{4,\parallel}^R = -D\}$ due to the continuity of the flow and the boundedness of $|x_4^R|$ on the section $\{x_{4,\parallel}^R = -2\}$ as assumed. We have

$$(6.32) \quad x_{4,\perp}(t) = x_{4,\perp}(t_0) + v_{4,\perp}(t_0)(t - t_0) + \int_{t_0}^t \int_{t_0}^u \ddot{x}_{4,\perp}(s) ds du,$$

where we have $\ddot{x}_{4,\perp}(s) = O\left(\frac{x_{4,\perp}(s)}{|x_4(s)|^3} + \frac{|x_4|}{\chi^3}\right) = O\left(\frac{x_{4,\perp}(s)}{(s+D)^3} + \frac{|x_4|}{\chi^3}\right)$. We bound $|x_{4,\perp}(s)|$ in the double integral by $|2(x_{4,\perp}(t_0) + v_{4,\perp}(t_0)(s - t_0))|$ over time interval $s \in [0, \tau)$ for some τ , to get the following for $t < \tau$,

$$1 - \frac{2C}{D} \leq \frac{x_{4,\perp}(t)}{(|x_{4,\perp}(t_0)| + |v_{4,\perp}(t_0)|(t - t_0))} \leq 1 + \frac{2C}{D}.$$

This shows that τ can be as large as order χ provided $D > 2C$. For large enough D , the slope $(1 \pm 2C/D)|v_{4,\perp}(t_0)|$ is bounded by $O(\mu\mathcal{G}/\chi)$ since we have estimates of $x_{1,\perp} = O(\mu\mathcal{G})$ on the section $\{x_{4,\parallel}^R = -\chi/2\}$. This shows $x_4^R \in \mathcal{T}_{G_0, \hat{C}}$ for the piece of orbit in consideration.

Step 5, bounding the left piece of the orbit x_4^L and the returning orbit.

Since the eccentricity e_4 gets close to 1 when Q_4 gets close to Q_1 as we argued on Step 3, we have $O(1)$ bound for $x_{4,\perp}^L$ on the section $\{x_{4,\parallel}^L = D\}$. We apply the same argument as on Step 4 to the left piece between the section $\{x_{4,\parallel}^L = D\}$ and section $\{x_{4,\parallel}^R = -\chi/2\}$. This shows $-x_4^L \in \mathcal{T}_{G_0, \hat{C}}$ for the piece of orbit in consideration.

For x_1^L, v_1^L , we first establish the estimates $v_1^L = O(1, \mathcal{G}/\chi)$, $x_1^L \in \mathcal{T}_{G_0, \hat{C}}$ on the section $\{x_{4,\parallel}^R = -\chi/2\}$ using (6.19), the estimate on $x_1^R, v_1^R, x_{4,\perp}^{L,R}$ and $v_{4,\perp}^{L,R}$ in Step 2. Then we integrate the \dot{x}_1, \dot{v}_1 equations.

For the returning orbit, the argument is similar except that we use the time reversed orbit.

The proof is now complete. \square

The next lemma is used to verify part of the assumption (6.6).

Lemma 6.7. *Suppose the initial orbit parameters $(L_3, \ell_3, G_3, g_3; x_1, v_1; G_4, g_4)$ for the local map satisfy **AL**. Then for μ sufficiently small and χ sufficiently large we have $\delta < |x_3| \leq 2 - \delta$ where $\delta > 0$ is a constant independent of μ and χ .*

Proof. We use Lemma 2.2 to get that the orbit of x_3, v_3 is a $o(1)$ small deformation of Gerver's Q_3 ellipse as $\mu, \tilde{\theta} \rightarrow 0$. So we only need to prove this lemma in Gerver's setting. Since the Q_3 ellipse has semimajor 1 in Gerver's case, the distance from the apogee to the focus is strictly less than 2. Therefore we can find some $\delta > 0$ such that $|Q_3| \leq 2 - 2\delta$ in the Gerver case. Next we know from the Sublemma 6.6 and its proof that x_4 moves away almost linearly (the oscillation of v_4 is small). We integrate the $\frac{dL_3}{d\ell_4}$ equation to get that the oscillation of L_3 is $O(\mu)$. \square

6.6. Collision exclusion. The following lemma excludes the possibility of collisions between Q_1 and Q_4 .

Lemma 6.8. *If the global map \mathbb{G} satisfies **AG** and there is a collision between Q_4 and Q_1 , then we have $\bar{G}_4^R + G_4^R = O(\mu)$ when evaluated on the section $\{x_{4,\parallel}^R = -2\}$, where G_4^R and \bar{G}_4^R are the angular momentums of $(x_4, v_4)^R$ before and after the application of the global map respectively.*

Proof. The assumption **AG** implies the assumptions of Lemma 6.2 according to Lemma 6.5 so we can use the conclusions of Lemma 6.2.

Suppose we have a collision. We compare the bouncing back orbit (subscript *out*) with the time reversal of the incoming orbit (subscript *in*). We will show that the orbits are close and so the values of G_e will be close. This will be achieved in several steps.

Step 1, Comparing orbits to the left of the line $x_{4,\parallel}^R = -\chi/2$.

Denote $\mathbf{Y} = (x_1, v_1; L_4, G_4, g_4)$ and let \mathbf{F} be the RHS of the corresponding Hamiltonian equations (6.3). We have

$$(\mathbf{Y}_{in} - \mathbf{Y}_{out})' = \frac{\partial \mathbf{F}}{\partial \mathbf{Y}}(\mathbf{Y}_{in})(\mathbf{Y}_{in} - \mathbf{Y}_{out}) + O(|\mathbf{Y}_{in} - \mathbf{Y}_{out}|^2) + O\left(\frac{\mu}{|x_4|^3} + \frac{1}{\chi^4}\right)$$

where the last parenthesis contains the terms involving x_3 which are estimated in parts (2),(3) and (4) of Lemma 6.1.

We denote by ℓ_4^i the initial time corresponding to the collision and ℓ_4^f be the time when the time reversed incoming orbit hits $\{x_{4,\parallel}^R = -\chi/2\}$.

Note the initial condition $(\mathbf{Y}_{in} - \mathbf{Y}_{out})(\ell_4^i) = 0$ and that the fundamental solution of the variational equation $Z' = \frac{\partial \mathbf{F}}{\partial \mathbf{Y}}Z$ is $O(\mu\chi)$ (the fundamental solution is given by the matrix M in Proposition 5.1). Since

$$O(\mu\chi) \int_{\ell_4^i}^{\ell_4^f} O\left(\frac{\mu}{|x_4 - x_3|^3}\right) d\ell_4 = O\left(\frac{\mu^2}{\chi}\right)$$

The Gronwell inequality gives

$$\mathbf{Y}_{in} - \mathbf{Y}_{out} = O\left(\frac{\mu^2}{\chi}\right).$$

Step 2, Cartesian coordinates. We already proved in Step 1 that $\delta(x_1^L, v_1^L) = O(\mu^2/\chi)$. We need to control the change of x_4 as well. We have

$$\delta(v_4, x_4)^L = \left[\delta L_4 \frac{\partial}{\partial L_4} + \delta \ell_4 \frac{\partial}{\partial \ell_4} + \delta G_4 \frac{\partial}{\partial G_4} + \delta g_4 \frac{\partial}{\partial g_4} \right]^L (v_4, x_4)^L.$$

We use the formulas (A.6) in the appendix to see that the partial derivatives which are $O(\chi)$. The estimates for $(\delta L_4, \delta G_4, \delta g_4)^L$ are obtained in Step 1.

To estimates $\delta \ell_4^L$ observe that the choice of section $x_{4,\parallel}^R = -\chi/2$ gives $\frac{1}{1+\mu} x_{4,\parallel}^L - x_{1,\parallel}^L = -\chi/2$. using the formula for $L \cdot R^{-1}$ from Proposition 5.1. So we have

$$\delta \ell_4^L = \frac{-1}{\frac{\partial x_{4,\parallel}^L}{\partial \ell_4}} \left(\left[\delta L_4 \frac{\partial}{\partial L_4} + \delta G_4 \frac{\partial}{\partial G_4} + \delta g_4 \frac{\partial}{\partial g_4} \right]^L x_4^L + (1+\mu) \delta x_{1,\parallel}^L \right) = O(\mu^2)$$

where the last estimate uses the $O(\mu^2/\chi)$ estimates for $(\delta L_4, \delta G_4, \delta g_4, \delta x_1)^L$, the $O(\chi)$ estimates for the partial derivatives and the $O(1)$ estimate for $\frac{\partial x_{4,\parallel}^L}{\partial \ell_4}$. This tells us that

$$\delta(x_4, v_4)^L = O(\mu^2, \mu^2, \mu^2/\chi, \mu^2/\chi).$$

Using $R \cdot L^{-1}$ from Proposition 5.1 we get

$$\delta(x_4, v_4)^R = O(\mu^2, \mu^2, \mu^2/\chi, \mu^2/\chi).$$

Step 3, Comparing angular momenta. Using the relation

$$\begin{aligned} (6.33) \quad G_4^R &= v_4^R \times x_4^R = \left(\frac{1+\mu}{1+2\mu} v_4^L + \frac{2\mu}{1+2\mu} v_1^L \right) \times \left(\frac{x_4^L}{1+\mu} + x_1^L \right) \\ &= \frac{G_4^L}{1+2\mu} + \frac{1+\mu}{1+2\mu} v_4^L \times x_1^L + \frac{2\mu}{(1+2\mu)(1+\mu)} v_1^L \times x_4^L + \frac{2\mu}{1+2\mu} v_1^L \times x_1^L \end{aligned}$$

and the results of Step 2 we get $\delta G_4^R = O(\mu^2)$.

Step 4, Conservation of G_4^R to the right of $\{x_{4,\parallel}^R = -\chi/2\}$. Now we consider the right pieces of orbits. We already saw in Section 6.4 that for the incoming orbit the oscillations of G_4^R are $O(\mu)$. Now we show the same result for the ejected orbit. Fix a large constant D . Let $\hat{\tau}$ be the first time when either $|x_4^R| = D$ or E_4^R or G_4^R change by at least 1% of their values at the section $\{x_{4,\parallel}^R = -\chi/2\}$. Let t_0 be the time the orbits visits this section and let ρ, ϕ denote the polar coordinates for x_4^R . Then for $\hat{\tau} \leq t$ we get

$$\dot{G}_4^R, \dot{E}_4^R = O\left(\frac{1}{\chi^2} + \frac{\mu}{r^2}\right)$$

and since

$$E_4^R = \frac{(\dot{r})^2 + r^2(\dot{\phi})^2}{2} + O\left(\frac{1}{r}\right) = \frac{(\dot{r})^2 + (G^R)^2/r^2}{2} + O\left(\frac{1}{r}\right) = \frac{(\dot{r})^2}{2} + O\left(\frac{1}{r}\right)$$

it follows that r decreases linearly. Accordingly for $t \leq \hat{\tau}$ the oscillations of both G_4 and E_4 are

$$O\left(\int_{t_0}^t \left(\frac{1}{\chi^2} + \frac{\mu}{r^2}\right) dt\right) = O\left(\frac{\mu}{D}\right).$$

Therefore $x_4^R(\hat{\tau}) = D$ and $G_4^R(\hat{\tau}) - G_4^R(t_0) = O(\mu)$. Also the oscillation of G_4^R after time $\hat{\tau}$ are $O(\mu)$ since the motion is a small perturbation of the Kepler motion completing the proof of Step 4.

Steps 1–4 show that difference between the angular momenta of reversed incoming orbit and the ejected orbit is $O(\mu)$. Without the time reversal we have $\bar{G}_4^R + G_4^R = O(\mu)$ as claimed. \square

The possibility of collision between Q_4 and Q_1 is excluded since in Gerver's construction, $\bar{G}_4 + G_4$ is always bounded away from zero independent of μ . Now we exclude the possibility of collisions between Q_3 and Q_4 . Note that Q_3 and Q_4 have two potential collision points corresponding to two intersections of the ellipse of Q_3 and the branch of the hyperbola utilized by Q_4 . See Fig 1 and 2. Now it follows from Lemma 10.1(b) that Q_3 and Q_4 do not collide near the intersection where they have the close encounter. We need also to rule out the collision near the second intersection point. This was done by Gerver in [G2]. Namely he shows that the time for Q_3 and Q_4 to move from one crossing point to the other are different. As a result, if Q_3 and Q_4 come to the correct intersection points nearly simultaneously, they do not collide at the wrong points. To see that the travel times are different recall that by second Kepler's law the area swiped by the moving body in unit time is a constant for the two-body problem. In terms of Delaunay coordinates, this fact is given by the equation $\dot{\ell} = \pm \frac{1}{L^3}$ where $-$ is for hyperbolic motion and $+$ for elliptic. In our case, we have $L_3 \approx L_4$ when $\mu \ll 1, \chi \gg 1$. Therefore in order to collide Q_3 and Q_4 must swipe nearly the same area within the unit time. We see from Fig 1 and Fig 2, the area swiped by Q_4 is a proper subset of that by Q_3 between the two crossing points. Therefore the travel time for Q_4 is shorter.

6.7. Proofs of Lemma 2.3 and 2.4. In this section, we prove of Lemma 2.3 and 2.4. We first show some finer estimates of the slope of the asymptotes in the next lemma, which justifies the assumption of Lemma A.2 in Appendix A and proves part (b) of Lemma 2.3.

Lemma 6.9. *Assume (6.6), (6.9), (6.25) (same assumptions as in Corollary 6.2). Then*

- (a) *when x_4 is moving to the right of the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$, we have*

$$\tan g_4 = \text{sign}(u) \frac{G_4}{L_4} + O\left(\frac{\mu}{|\ell_4|^2 + 1} + \frac{\mu \mathcal{G}}{\chi}\right), \quad \text{as } |\ell_4|, \chi \rightarrow \infty, \mu \rightarrow 0.$$

- (b) *When x_4 is moving to the left of the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$, then $G_4, g_4 = O(\mu \mathcal{G}/\chi)$ as $\chi \rightarrow \infty$ and $\mu \rightarrow 0$.*

Proof. Step 1. We prove part (b). Integrating the Hamiltonian equation for G_4^L, g_4^L in Corollary 6.2 starting from $\ell_4 = 0$ we get $(G_4^L, g_4^L)(\ell_4) = (G_4^L, g_4^L)(0) + O(\mu \mathcal{G}/\chi)$ when arriving at the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$. To conclude part (b), we need to show the initial conditions $G_4^L(0), g_4^L(0)$ are bounded by $O(\mu \mathcal{G}/\chi)$. Using (A.6) (we omit the superscript L and subscript 4), we have on the sections

$$\left\{x_{4,\parallel}^R = -\chi/2\right\} \text{ and } \left\{x_{4,\parallel}^L = \chi/2\right\}$$

$$\begin{aligned} x_{4,\perp} &= \frac{1}{mk}(\sin g L^2(\cosh u - e) + \cos g L G \sinh u) \\ &= \frac{1}{mk}(\sin g(0) L^2(\cosh u - e) + \cos g(0) L G(0) \sinh u) + O(\mu \mathcal{G}). \end{aligned}$$

Note that this holds for both large positive and large negative u and that on both sections $\cosh u$ and $|\sinh u|$ are of order χ . The assumption (6.6) shows that $|x_{4,\perp}| \leq 2\hat{C}(1 + \mu \mathcal{G})$ on the sections $\left\{x_{4,\parallel}^R = -\chi/2\right\}$ and $\left\{x_{4,\parallel}^L = \chi/2\right\}$. This implies that $|g(0)|, |G(0)| = O(\mu \mathcal{G}/\chi)$.

Step 2. Then we use the matrix $R \cdot L^{-1}$ in Proposition 5.1 to convert the left variables to the right to obtain $v_4^R = O(\mu)v_1^L \pm (1 + O(\mu))v_4^L$. From Step 1 and (A.6), we get the slope of v_4^L is $g_4^L - \arctan \frac{G_4^L}{L_4^L} + O(1/\chi^2) = O(\mu \mathcal{G}/\chi)$, and from part (c) of Lemma 6.2 the slope of v_1^L is $O(\mu \mathcal{G}/\chi)$. So the slope of v_4^R is $g_4^R - \arctan \frac{G_4^R}{L_4^R} + O(1/\chi^2) = O(\mu \mathcal{G}/\chi)$ on the sections $\left\{x_{4,\parallel}^R = -\chi/2\right\}$ and $\left\{x_{4,\parallel}^L = \chi/2\right\}$ due to (A.5).

Step 3. To prove part (a), we use the $O(\mu \mathcal{G}/\chi)$ estimates of the slope of v_4^R in Step 2 as initial condition. We get that the oscillation of G_4^R, g_4^R, L_4^R is $O\left(\frac{\mu}{|\ell_4|^2 + 1} + \frac{\mu \mathcal{G}}{\chi}\right)$ from $\ell_4 = O(\chi)$ to ℓ_4 by integrating the $\frac{dG_4^R}{d\ell_4}, \frac{dg_4^R}{d\ell_4}$ estimates in Corollary 6.2 together with the same estimate of $\frac{dL_4^R}{d\ell_4}$ obtained directly from the Hamiltonian using $\frac{\partial x_4^R}{\partial \ell_4^R} = O(1)$. \square

Next, we prove Lemma 2.3.

Proof of Lemma 2.3. The assumption of this lemma implies those of Lemma 6.2 due to Lemma 6.7 and 6.5.

The part (a) is done by integrating the estimates of the Hamiltonian equations for $\frac{dL_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dg_3}{d\ell_4}$ in Corollary 6.2 over time of order χ .

Part (b) follows directly from Lemma 6.9. \square

Now we are ready to prove Lemma 2.4.

Proof of Lemma 2.4. Step 1, bounding the horizontal velocity $\mathcal{R}(\bar{v}_{1,\parallel})$.

The argument in this step is a refinement of part of the proof of Lemma 6.2.

Substep 1.0, preparations.

We begin with $v_{1,\parallel}$ estimate. Without loss of generality, we consider only the more difficult case $\mathcal{G}_\chi = \varepsilon_0 \sqrt{\chi}$. We use (6.19) and (6.23) together with the estimate

$$\left| \frac{dv_{1,\parallel}}{dt} \right| = \left| \frac{k_1 x_{1,\parallel}}{|x_1|^3} + h.o.t. \right| \leq \frac{2}{\mu \chi^2}, \quad \frac{dx_{1,\parallel}}{dt} = v_{1,\parallel}/m_1$$

coming from (6.3), to get that $v_1 = v_1(0) + O(1/\mu\chi, \mathcal{G}_\chi/\chi^2)$ during time $O(\chi)$. It follows from $\frac{dx_{1,\parallel}}{dt} = v_{1,\parallel}/m_1$ that over time $O(\chi)$, the horizontal component $x_{1,\parallel}$ can move only distance $O(\mu\chi)$. Moreover, the local map takes only $O(1)$ time as $1/\chi \ll \mu \rightarrow 0$.

Initially, we have angular momentum conservation $G_1 + G_3 + G_4 = G$. Also

$$|G_3|, |G_4| \leq 2(C_4 + 2),$$

due to assumptions (i), (ii), and (iv) of the lemma and the vanishing of energy. We get from the definition of angular momentum, Lemma 6.2(c) and assumption (iii)

$$|v_{1,\perp}(0)| \leq \left| v_{1,\parallel} \frac{x_{1,\perp}}{x_{1,\parallel}} \right| + \left| \frac{G_1}{x_{1,\parallel}} \right| \leq O\left(\frac{\mu}{\chi^{3/2}}\right) + \frac{\mathcal{G}_\chi + 4(C_4 + 2)}{\chi} \leq 5(C_1 + 2) \frac{\mathcal{G}_\chi}{\chi}.$$

Substep 1.1, piece (I) composed with the local map.

We integrate the $\frac{dv_{1,\parallel}}{dt}$ estimates from the section $\{x_{4,\parallel}^R = -2, v_{4,\parallel}^R > 0\}$ to the section $\{x_{4,\parallel}^R = -\chi/2, v_{4,\parallel}^R < 0\}$ (notice that the local map is included). The time is in between $\chi/(3\bar{c}_1)$ and χ/c_1 since the total distance is $(1 + O(\mu))\chi/2$ and the velocity satisfies $v_{4,\parallel} \in [-\bar{c}_1, -c_1]$ up to $O(1/\mu\chi)$ error. In the following we use the notation $a = O_+(b)$ if $a = O(b)$ and $\frac{a}{b} > c > 0$ for some constant c . Using (6.1) we have as $1/\chi \ll \mu \rightarrow 0$,

$$\begin{aligned} x_{1,\parallel}^R - x_{1,\parallel}^R(0) &= O_+(\mu\chi)v_{1,\parallel}^R(0), \quad x_{1,\perp}^R - x_{1,\perp}^R(0) = O(\mu\chi)v_{1,\perp}^R(0) = O(\mu\mathcal{G}_\chi), \\ v_{1,\parallel}^R &\in [-\bar{c}_1, -c_1] + O(1/\mu\chi), \quad v_{1,\perp}^R = O(\mathcal{G}_\chi/\chi). \end{aligned}$$

on the section $\{x_{4,\parallel}^R = -\chi/2, v_{4,\parallel}^R < 0\}$. On the same section, we also have

$$v_{4,\parallel}^R = -\sqrt{2E_4} + O(\mu\mathcal{G}_\chi/\chi) = -\sqrt{-2E_3} + O(\bar{c}_1^2\mu) + O(\mu\mathcal{G}_\chi/\chi), \quad v_{4,\perp}^R = O(\mu\mathcal{G}_\chi/\chi).$$

where the first “=” comes from Lemma 6.9 and (A.5), and the second “=” comes from the energy conservation.

Substep 1.2, piece (III).

We use Lemma 6.5 to get $x_4^R \in \mathcal{T}_{G_0, \hat{C}}$, i.e. $|x_{4,\parallel}^R| \leq 2\chi$, $|x_{4,\perp}^R| \leq \hat{C} + \hat{C}\mu\mathcal{G}$. Then we use (6.19) to get on the section $\{x_{4,\parallel}^R = -\chi/2, v_{4,\parallel}^R < 0\}$

(6.34)

$$x_{1,\parallel}^L = \frac{1}{1+\mu}x_{1,\parallel}^R(0) + O_+(\mu\chi)v_{1,\parallel}^R(0) - \frac{\mu\chi}{1+2\mu} = \frac{1}{1+\mu}x_{1,\parallel}^R(0) - \frac{\mu\chi}{1+2\mu} - O_+(\mu\chi),$$

$$x_{1,\perp}^L = \frac{1}{1+\mu}x_{1,\perp}^R(0) + O(\mu\chi)v_{1,\perp}^R(0) + O(\mu^2\mathcal{G}) = O(\mu\mathcal{G}),$$

$$v_{1,\parallel}^L \in \frac{1+\mu}{1+2\mu}[-\bar{c}_1, -c_1] - \sqrt{-2E_3} + O(\mu\mathcal{G}_\chi/\chi), \quad v_{1,\perp}^L = O(\mathcal{G}_\chi/\chi), \quad \text{as } 1/\chi \ll \mu \rightarrow 0.$$

We integrate $\frac{dv_1}{dt}$ again over time in between $\frac{\chi}{2(\bar{c}_1 + \sqrt{-2E_3})}$ and $2\chi/c_1$ to get

(6.35)

$$\begin{aligned} x_{1,\parallel}^L &= \frac{1}{1+\mu} x_{1,\parallel}^R(0) - \frac{\mu\chi}{1+2\mu} - O_+(\mu\chi), \quad x_{1,\perp}^L = O(\mu\mathcal{G}_\chi), \\ v_{1,\parallel}^L &\in \frac{1+\mu}{1+2\mu} [-\bar{c}_1, -c_1] - \sqrt{-2E_3} + O(\mu\mathcal{G}_\chi/\chi), \quad v_{1,\perp}^L = O(\mathcal{G}_\chi/\chi), \text{ as } \frac{1}{\chi} \ll \mu \rightarrow 0 \end{aligned}$$

when arriving at the section $\{x_{4,\parallel}^L = \chi/2\}$ where the $-O_+(\mu\chi)$ term in $x_{1,\parallel}^L$ has absorbed a new $-O_+(\mu\chi)$ contribution since $v_{1,\parallel}^L < 0$. Again it follows from Lemma 6.9 and the energy conservation that

$$v_{4,\parallel}^L = -\sqrt{2E_4} + O(\mu\mathcal{G}_\chi/\chi) = -\sqrt{-2E_3} + O(\mu)\bar{c}_1^2 + O(\mu\mathcal{G}_\chi/\chi), \quad v_{4,\perp}^R = O(\mu\mathcal{G}_\chi/\chi).$$

Substep 1.3, piece (V).

We then apply (6.23) and $-x_4^L \in \mathcal{T}_{\mathcal{G}_0, \hat{c}}$ (Lemma 6.5) to get on the section $\{x_{4,\parallel}^L = \chi/2\}$,

$$\begin{aligned} (6.36) \quad x_{1,\parallel}^R &= \frac{1+\mu}{1+2\mu} \left(\frac{x_{1,\parallel}^R(0)}{1+\mu} - \frac{\mu\chi}{1+2\mu} \right) - \frac{\mu\chi}{1+2\mu} - O_+(\mu\chi) \\ &= \frac{x_{1,\parallel}^R(0)}{1+2\mu} - \frac{\mu(2+3\mu)\chi}{(1+2\mu)^2} - O_+(\mu\chi), \\ x_{1,\perp}^R &= O(\mu\mathcal{G}_\chi), \quad v_{1,\perp}^R = O(\mathcal{G}_\chi/\chi), \\ v_{1,\parallel}^R &\in \frac{1}{1+2\mu} [-\bar{c}_1, -c_1] - 2\sqrt{-2E_3^*} + O(\delta + \mu + \mu\mathcal{G}_\chi/\chi), \end{aligned}$$

as $1/\chi \ll \mu \rightarrow 0$, where the extra $O(\mu)$ in $v_{1,\parallel}^R$ comes from the oscillation of E_3 established in Lemma 2.3(a), and the $O(\delta)$ is the deviation of initial value E_3 from Gerver's value E_3^* , which is bounded by $C_3\delta$. Finally, we get the same estimate as (6.36) when arriving at the section $\{x_{4,\parallel}^R = -2, v_{4,\parallel}^R > 0\}$ with a new $-O_+(\mu\chi)$ added to $x_{1,\parallel}^R$. This completes one application of \mathcal{P} . The information that we need from $x_{1,\parallel}^R$ is that $x_{1,\parallel}^R < x_{1,\parallel}^R(0)$ after one application of \mathcal{P} . Indeed, it follows from the first row of (6.36) and the assumption on $x_{1,\parallel}^R$ that

(6.37)

$$x_{1,\parallel}^R - x_{1,\parallel}^R(0) = \frac{2\mu\chi}{1+2\mu} - \frac{\mu(2+3\mu)\chi}{(1+2\mu)^2} + O(\chi^{-1/2}) - O_+(\mu\chi) = O(\mu^2\chi) - O_+(\mu\chi) < 0.$$

Substep 1.4, renormalization.

Since one period in Gerver's construction consists of $\mathcal{R} \circ \mathcal{P}^2$. We repeat the above procedure to get after \mathcal{P}^2

$$\begin{aligned} (6.38) \quad x_{1,\parallel}^R - x_{1,\parallel}^R(0) &= -O_+(\mu\chi) < 0, \quad x_{1,\perp}^R = O(\mu\mathcal{G}_\chi), \quad v_{1,\perp}^R = O(\mathcal{G}_\chi/\chi) \\ \bar{v}_{1,\parallel}^R &\in \frac{1}{(1+2\mu)^2} [-\bar{c}_1, -c_1] - 2\sqrt{-2E_3^*} - 2\sqrt{-2E_3^{**}} + O(\delta + \mu + \mu\mathcal{G}_\chi/\chi), \end{aligned}$$

as $1/\chi \ll \mu \ll \delta \rightarrow 0$, The last step is to apply the renormalization \mathcal{R} . Let us forget about the rotation by β in Definition 2.4 for a moment and consider only the

rescaling. We expect that

$$\mathcal{R}(\bar{v}_{1,\parallel}^R) = \frac{1}{\sqrt{\lambda}} \bar{v}_{1,\parallel}^R \in [-\bar{c}_1, -c_1],$$

which is implied by

$$\bar{v}_{1,\parallel}^R \in \frac{1}{(1+2\mu)^2} [-\bar{c}_1, -c_1] - 2\sqrt{-2E_3^*} - 2\sqrt{-2E_3^{**}} + O(\delta + \mu + \mu\mathcal{G}_\chi/\chi) \subset \sqrt{\lambda} [-\bar{c}_1, -c_1],$$

where λ is the renormalization factor in Definition 2.4. This implies

$$c_1 + \tilde{c}(\delta + \mu + \mu\mathcal{G}_\chi/\chi) \leq \frac{2}{\sqrt{\lambda}-1} (\sqrt{-2E_3^*} + \sqrt{-2E_3^{**}}) \leq \bar{c}_1 - \tilde{c}(\delta + \mu + \mu\mathcal{G}_\chi/\chi)$$

for some constant \tilde{c} bounding the O in the above estimates. We choose

$$\bar{c}_1 = \frac{4}{\sqrt{\lambda}-1} (\sqrt{-2E_3^*} + \sqrt{-2E_3^{**}}), \quad c_1 = \frac{1}{\sqrt{\lambda}-1} (\sqrt{-2E_3^*} + \sqrt{-2E_3^{**}})$$

so that the above inequality is satisfied uniformly for all small enough $\mu, \delta, 1/\chi$. This completes the proof for $\mathcal{R}(\bar{v}_{1,\parallel}^R)$.

Step 2, part (b), the estimates of $\mathcal{R}(x_1)$.

This estimate follows by iterating (6.37) twice and applying the renormalization map \mathcal{R} .

Now let us take care of the rotation β of $\mathcal{R}(\bar{v}_{1,\parallel})$. This produces an error of $O(\mu\mathcal{G}_\chi/\chi)$ to $\bar{v}_{1,\parallel}$, which can be absorbed into the O part of the estimate of $\bar{v}_{1,\parallel}$ in (6.38) so that we leave our choice of c_1, \bar{c}_1 unchanged.

Step 3, bounding the angular momentum and vertical component of the velocity $\bar{v}_{1,\perp}$.

After \mathcal{P}^2 and $\mathcal{R} \circ \mathcal{P}^2$, we have angular momentum conservation

$$\bar{G}_1 + \bar{G}_3 + \bar{G}_4 = \bar{G} = G, \text{ and } |\mathcal{R}(\bar{G}_1) + \mathcal{R}(\bar{G}_3) + \mathcal{R}(\bar{G}_4)| = |\mathcal{R}(\bar{G})| = |\sqrt{\lambda}G| \leq \sqrt{\lambda}\mathcal{G}_\chi \leq \mathcal{G}_{\tilde{\chi}}$$

respectively. After renormalization $\mathcal{R}(E_3)$ is now $-1/2$. The energy conservation shows that $|\mathcal{R}(v_4)| \leq 1 + O(\mu)$, so that we have

$$|\mathcal{R}(\bar{G}_3)|, |\mathcal{R}(\bar{G}_4)| \leq 2(\lambda C_4 + 2)$$

using assumption (i), (ii) in the lemma. So we get according to assumption (iii)

$$|\mathcal{R}(\bar{G}_1)| \leq |\mathcal{R}(\bar{G})| + |\mathcal{R}(\bar{G}_3)| + |\mathcal{R}(\bar{G}_4)| \leq \mathcal{G}_{\tilde{\chi}} + 4(\lambda C_4 + 2).$$

We get from the definition of angular momentum and (6.38) that

$$|\mathcal{R}(\bar{v}_{1,\perp})| \leq \left| \mathcal{R}(\bar{v}_{1,\parallel}^R) \frac{\mathcal{R}(\bar{x}_{1,\perp}^R)}{\mathcal{R}(\bar{x}_{1,\parallel}^R)} \right| + \left| \frac{\mathcal{R}(\bar{G}_1)}{\mathcal{R}(\bar{x}_{1,\parallel}^R)} \right| \leq O\left(\frac{\mu\mathcal{G}_\chi}{\chi}\right) + \frac{\mathcal{G}_{\tilde{\chi}} + 4(\lambda C_4 + 2)}{\tilde{\chi}} \leq 5(\lambda C_4 + 2) \frac{\mathcal{G}_{\tilde{\chi}}}{\tilde{\chi}}.$$

This completes the proof of the $\mathcal{R}(\bar{v}_{1,\perp})$ estimate in part (c) by defining $C_1 := 5(\lambda C_4 + 2)$. \square

6.8. Choosing angular momentum, proof of Lemma 2.7. In this section, we prove Lemma 2.7. We first need two auxiliary results.

Sublemma 6.10. *Let \tilde{e}_4 be that in part (a) of Lemma 2.7, then there exists $\tilde{\ell}_3$ such that $\mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) \in U_2(\delta)$.*

We give the proof of the sub lemma immediately we complete the proof of Lemma 2.7.

Sublemma 6.11. *Let F be a map on \mathbb{R}^2 which fixes the origin and such that if $|F(z)| < R$ then $\|dF(X)\| \geq \bar{\chi}\|X\|$. Then for each a such that $|a| < R$ there exists z such that $|z| < R/\bar{\chi}$ and $F(z) = a$.*

Proof. Without the loss of generality we may assume that $a = (r, 0)$. Let $V(z)$ be the direction field defined by the condition that the direction of $dF(V(z))$ is parallel to $(1, 0)$. Let $\gamma(t)$ be the integral curve of V passing through the origin and parameterized by the arclength. Then $F(\gamma(t))$ has form $(\sigma(t), 0)$ where $\sigma(0) = 0$ and $|\dot{\sigma}(t)| \geq \bar{\chi}$ as long as $|\sigma| < R$. Now the statement follows easily. \square

With the help of the two sub lemmas, we finish the proof of Lemma 2.7.

Proof of Lemma 2.7. (a) We claim that it suffices to show that for each $(\bar{e}_4, \bar{\ell}_3)$ such that $|\bar{e}_4 - e_4^{**}| < \sqrt{\delta}$ there exist $(\hat{e}_4, \hat{\ell}_3)$ such that

$$(6.39) \quad \mathcal{P}(\hat{e}_4, \hat{\ell}_3) = (\bar{e}_4, \bar{\ell}_3).$$

Indeed in that case Sublemma 6.6 says that the outgoing asymptote is almost horizontal. Therefore by Lemma 2.2 our orbit has (E_3, e_3, g_3) close to

$\mathbf{G}_{\bar{e}_4, 2, 4}(E_3(\hat{e}_4, \hat{\ell}_3), e_3(\hat{e}_4, \hat{\ell}_3), g_3(\hat{e}_4, \hat{\ell}_3))$. Next Lemma 2.3 shows that after the application of \mathbb{G} , (E_3, e_3, g_3) change little and θ_4^- becomes $O(\mu)$ so that $\mathcal{P}(\hat{e}_4, \hat{\ell}_3) \in U_2(K\delta)$.

We will now prove (6.39). Our coordinates allow us to treat \mathcal{P} as a map $\mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R} \times \mathbb{T}$. We first apply Sublemma 6.10 to find $\tilde{\ell}_3$ for each \tilde{e}_4 such that $\mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) \in U_2(\delta)$. Due to Lemma 2.5 we can apply Sublemma 6.11 to the covering map $\tilde{\mathcal{P}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\bar{\chi} = c\chi$ obtaining (6.39). Part (b) of the lemma is similarly proven.

Part (c) is part of Lemma 10.1 to be proven later in Section 10. \square

Proof of the Sublemma 6.10. The idea is to apply the strong expansion of the Poincaré map in a neighborhood of the collisional orbit studied in Lemma 6.8. Notice Delaunay coordinates regularizes double collisions in the sense that none of the variables blows up at double collision, so that our estimate of $d\mathbb{G}$ holds also for collisional orbits.

Step 1. We first show that there is a collisional orbit satisfying $x_4^L(t) = 0$ at some time t as ℓ_3 varies. The proof of Lemma 6.8 shows that x_4^R nearly returns to its initial position. Sublemma 6.6 shows that if after the application of the local map we have $\theta_4^+(0) = \pi - \bar{\theta}$, $0 < \bar{\theta} < (\bar{\theta}$ in Lemma 3.1), then the orbit hits the section $\{x_{4,\parallel}^R = -\chi/2\}$ with the $x_{4,\perp}^R$ coordinate being a large positive number of order $\bar{\theta}\chi/2$. We know from the proof of Sublemma 6.6 that the orbit of $x_4^{L,R}$ moves approximately linearly. Next we convert to the left variables using $L \cdot R^{-1}$ so that the $x_{4,\perp}^L$ coordinate is a large positive number of order $\bar{\theta}\chi$ when the orbit hits $\{x_{4,\parallel}^L = 0\}$. Similarly, if $\theta_4^+(0) = \pi + \bar{\theta}$ then the orbit hits the line $\{x_{4,\parallel}^L = 0\}$ so that its $x_{4,\perp}^L$ coordinate is a large negative number. Therefore due to the Intermediate Value Theorem it suffices to show that our admissible surface S_j , $j = 1, 2$, contains points $\mathbf{x}_1, \mathbf{x}_2$ such that $\theta_4^+(\mathbf{x}_1) = \pi - \bar{\theta}$, $\theta_4^+(\mathbf{x}_2) = \pi + \bar{\theta}$. We have the expression $\theta_4^+ = g_4^+ - \arctan \frac{G_4^+}{L_4^+}$. By direct calculation we find $d\theta^+ = L_4^+ \hat{\mathbf{1}}$ (see also item

(2) of Remark 3.2). Since $TS_j \subset \mathcal{K}_j$ and the cone \mathcal{K}_j is centered at the plane $\text{span}\{\bar{\mathbf{u}}_{3-j}, \bar{\mathbf{u}}_{3-j}\}$. Note that $\bar{\mathbf{u}}_{3-j} \rightarrow \tilde{w} = \frac{\partial}{\partial \ell_3}$. We get using Lemma 3.4

$$d\theta^+ \cdot (d\mathbb{L}\bar{\mathbf{u}}_{3-j}) = L_4^+ \hat{\mathbf{l}}_j \cdot \left(\frac{1}{\mu} (\hat{\mathbf{u}}_j(\hat{\mathbf{l}}_j \tilde{w}) + o(1)) + O(1) \right) = c_j(\mathbf{x})/\mu, \quad c_j(\mathbf{x}_j) \neq 0.$$

So it is enough to vary ℓ_3 in a $O(\mu)$ neighborhood of a point whose outgoing asymptotes satisfies the assumption of Lemma 3.1.

Step 2. We show that there exists ℓ_3 such that $\bar{e}_4(\mathcal{P}(\ell_3, \tilde{e}_4))$ is close to e_4^{**} . We fix \tilde{e}_4 then \mathcal{P} becomes a function of one variable ℓ_3 . Suppose the collisional orbit in Step 1 occurs at $\ell_3 = \hat{\ell}_3$. As we vary ℓ_3 , the same calculation as in Step 1 gives $\hat{\mathbf{l}}_j \cdot (d\mathbb{L}\bar{\mathbf{u}}_{3-j}) = \bar{c}_j(\mathbf{x})/\mu$, $\bar{c}_j(\mathbf{x}_j) \neq 0$ and that $\bar{\mathbf{u}}_j$ contains nonzero $\partial/\partial e_4$ component. Therefore the projection of $\mathcal{P} = \mathbb{G} \circ \mathbb{L}$ to the e_4 component, i.e. $\bar{e}_4(\ell_3, \tilde{e}_4)$ as a function of ℓ_3 is strongly expanding with derivative bounded from below by $\bar{c}\chi^2/\mu$ provided that the assumptions of Lemma 6.2 are satisfied (for the orbits of interest this will always be the case according to Lemma 6.5). Since the map $\bar{e}_4(\ell_3, \tilde{e}_4)$ is not injective, we study $\bar{G}_4(\ell_3, \tilde{e}_4)$ instead of $\bar{e}_4(\ell_3, \tilde{e}_4)$ using the relation $e = \sqrt{1 + 2(G/L)^2}$. We have the same strong expansion for $\bar{G}_4(\ell_3, \tilde{e}_4)$ since our estimates of the $d\mathbb{L}, d\mathbb{G}$ are done using G_4 instead of e_4 . Thus it follows from the strong expansion of the map $\bar{G}_4(\ell_3, \tilde{e}_4)$ that a R -neighborhood of G_4^{**} (corresponding to e_4^{**}) is covered if ℓ_3 varies in a $\frac{R\mu}{\bar{c}\chi^2}$ -neighborhood of $\hat{\ell}_3$. Taking

R large we can ensure that \bar{G}_4 changes from a large negative number to a large positive number. Then we use the intermediate value theorem to find e_4 such that $|\bar{G}_4 - G_4^{**}| < \sqrt{\delta}$, hence $|\bar{e}_4 - e_4^{**}| < \sqrt{\delta}$.

Step 3. We show that for the orbit just constructed $\mathcal{P}(\tilde{\ell}_3, \tilde{e}_4) \in U_2(\delta)$. We apply Lemma 3.2 to get that $\theta_4^+ = O(\mu)$. Therefore by Lemma 2.2 $\mathbb{L}(\tilde{e}_4, \tilde{\ell}_3)$ has (E_3, e_3, g_3) close to $\mathbf{G}_{\tilde{e}_4, 2, 4}(E_3(\tilde{e}_4, \tilde{\ell}_3), e_3(\tilde{e}_4, \tilde{\ell}_3), g_3(\tilde{e}_4, \tilde{\ell}_3))$. It follows that

$$|E_3 - E_3^{**}| < K\delta, \quad |e_3 - e_3^{**}| < K\delta, \quad |g_3 - g_3^{**}| < K\delta.$$

Next Lemma 2.3 shows that after the application of \mathbb{G} , (E_3, e_3, g_3) change little and θ_4^- becomes $O(\mu)$. \square

7. THE VARIATIONAL EQUATION AND ITS SOLUTION

In this section, we first estimate the 10×10 matrices appearing in the variational equations. Next, we estimate the solution of the variational equations hence prove the N_1, M, N_5 in Proposition 5.1.

7.1. Estimates of the variational equation. Recall $u(\ell_4)$, $v(\ell_4)$ defined in Lemma 6.3 and Corollary 6.2 respectively and define further $w(\ell_4)$,

$$u(\ell_4) = \frac{1}{\chi^3} + \frac{\mu}{|\ell_4|^3 + 1}, \quad v(\ell_4) = \frac{\mu \mathcal{G}}{\chi^2} + \frac{\mu}{|\ell_4|^3 + 1}, \quad w(\ell_4) = \frac{1}{\chi} + \frac{\mu}{|\ell_4|^3 + 1}.$$

We start with the following auxiliary estimate.

Lemma 7.1. *Under the assumptions of Corollary 6.2 (that is, (6.6), (6.9), and (6.25)) we have the following estimates.*

$$\begin{aligned}
\text{(a)} \quad & \frac{dL_4^R}{d\mathcal{V}^R} = (1, 0_{1 \times 9}) + O\left(\mu, u(\ell_4), u(\ell_4), u(\ell_4); \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi}; v(\ell_4), v(\ell_4)\right). \\
& \frac{dL_4^L}{d\mathcal{V}^L} = (1, 0_{1 \times 9}) + O\left(\mu, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi}; \frac{\mu\mathcal{G}}{\chi^2}, \frac{\mu\mathcal{G}}{\chi^2}\right). \\
\text{(b)} \quad & \frac{d(dt/d\ell_4^R)}{d\mathcal{V}^R} = 3L_3^2(1, 0_{1 \times 9}) + O\left(\mu, (u(\ell_4))_{1 \times 3}; \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi}; w(\ell_4), w(\ell_4)\right). \\
& \frac{d(dt/d\ell_4^L)}{d\mathcal{V}^L} = 3L_3^2(1, 0_{1 \times 9}) + O\left(\mu, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi}; \frac{1}{\chi}, \frac{1}{\chi}\right).
\end{aligned}$$

Proof. Remember that $\frac{d}{d\mathcal{V}} = \frac{\partial}{\partial\mathcal{V}} + \frac{\partial L_4}{\partial\mathcal{V}} \frac{\partial}{\partial L_4}$. We use the same argument as the proof of Corollary 6.2 to estimate the angles among the vectors $x_4, x_1, \frac{\partial x_4}{\partial G_4}$, etc. Part (a) follows directly from equation (6.1). Part (b) follows from equation (6.2) and equation (6.1).

Part (a) and (b) differ only in their G_4, g_4 components. The estimates of the G_4, g_4 component in part (a) is the same as that of Corollary 6.2. However, for part (b) we have a $\frac{\partial U}{\partial L_4}$ term in (6.2). As a result we do not have the almost orthogonality of $\frac{\partial x_4}{\partial G_4}$ with x_1 as we did in the proof of Corollary 6.2. In the right case of part (b), there is also a contribution of order $\mu/(|\ell_4|^3 + 1)$ resulting in a worse estimate. \square

We also need to figure out the order of magnitude of each entry of the RHS of the variational equation.

Lemma 7.2. *Assume (6.6), (6.9), (6.25) (same assumptions as Corollary 6.2). Then*

(a) *in the right case, we have*

$$\frac{\partial \mathcal{F}^R}{\partial \mathcal{V}^R} \lesssim \begin{bmatrix}
u(\ell_4) & u(\ell_4)_{1 \times 3} & \frac{u(\ell_4)}{\mu\chi^2} & \frac{u(\ell_4)\mathcal{G}}{\chi^3} & \mu u(\ell_4) & \frac{\mu u(\ell_4)\mathcal{G}}{\chi} & \left(\frac{\mu}{\ell_4^3+1}\right)_{1 \times 2} \\
\mu & u(\ell_4)_{1 \times 3} & \frac{1}{\mu\chi^2} & \frac{\mathcal{G}}{\chi^3} & \mu & \frac{\mu\mathcal{G}}{\chi} & w(\ell_4)_{1 \times 2} \\
u(\ell_4) & u(\ell_4)_{1 \times 3} & \frac{u(\ell_4)}{\mu\chi^2} & \frac{u(\ell_4)\mathcal{G}}{\chi^3} & \mu u(\ell_4) & \frac{\mu u(\ell_4)\mathcal{G}}{\chi} & \left(\frac{\mu}{\ell_4^3+1}\right)_{1 \times 2} \\
u(\ell_4) & u(\ell_4)_{1 \times 3} & \frac{u(\ell_4)}{\mu\chi^2} & \frac{u(\ell_4)\mathcal{G}}{\chi^3} & \mu u(\ell_4) & \frac{\mu u(\ell_4)\mathcal{G}}{\chi} & \left(\frac{\mu}{\ell_4^3+1}\right)_{1 \times 2} \\
\hline
\mu & (u(\ell_4)\mu)_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu\mathcal{G}}{\chi^3} & \mu & \frac{\mu^2\mathcal{G}}{\chi} & (\mu w(\ell_4))_{1 \times 2} \\
\frac{\mu\mathcal{G}}{\chi} & \left(u(\ell_4)\frac{\mu\mathcal{G}}{\chi}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^3} & \frac{\mu\mathcal{G}^2}{\chi^4} & \frac{\mu^2\mathcal{G}}{\chi} & \mu & \left(\frac{\mu\mathcal{G}w(\ell_4)}{\chi}\right)_{1 \times 2} \\
\frac{1}{\mu\chi^2} & \left(\frac{u(\ell_4)}{\chi}\right)_{1 \times 3} & \frac{1}{\mu\chi^3} & \frac{\mathcal{G}}{\chi^4} & \frac{1}{\chi^2} & \frac{\mathcal{G}}{\chi^3} & \left(\frac{v(\ell_4)}{\chi}\right)_{1 \times 2} \\
\frac{1}{\chi^2} & \left(\frac{u(\ell_4)}{\chi}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^4} & \frac{1}{\mu\chi^3} & \frac{\mu}{\chi^2} & \frac{\mu\mathcal{G}}{\chi^3} & \left(\frac{v(\ell_4)}{\chi}\right)_{1 \times 2} \\
\hline
w(\ell_4) & \left(\frac{\mu}{|\ell_4|^3+1}\right)_{1 \times 3} & \frac{v(\ell_4)}{\chi} & \frac{v(\ell_4)}{\chi} & \mu w(\ell_4) & \frac{\mu w(\ell_4)\mathcal{G}}{\chi} & (w(\ell_4))_{1 \times 2} \\
w(\ell_4) & \left(\frac{\mu}{|\ell_4|^3+1}\right)_{1 \times 3} & \frac{v(\ell_4)}{\chi} & \frac{v(\ell_4)}{\chi} & \mu w(\ell_4) & \frac{\mu w(\ell_4)\mathcal{G}}{\chi} & (w(\ell_4))_{1 \times 2}
\end{bmatrix}$$

(b) in the left case, we have

$$\frac{\partial \mathcal{F}^L}{\partial \mathcal{V}^L} \lesssim \begin{bmatrix} \frac{1}{\chi^3} & \left(\frac{1}{\chi^3}\right)_{1 \times 3} & \frac{1}{\chi^4} & \frac{1}{\chi^4} & \frac{\mu}{\chi^3} & \frac{1}{\chi^4} & \left(\frac{\mu}{\chi^3}\right)_{1 \times 2} \\ \mu & \left(\frac{1}{\chi^3}\right)_{1 \times 3} & \frac{1}{\mu \chi^2} & \frac{1}{\chi^3} & \mu & \frac{\mu}{\chi^4} & \left(\frac{1}{\chi}\right)_{1 \times 2} \\ \frac{1}{\chi^3} & \left(\frac{1}{\chi^3}\right)_{1 \times 3} & \frac{1}{\chi^4} & \frac{1}{\chi^4} & \frac{\mu}{\chi^3} & \frac{1}{\chi^4} & \left(\frac{\mu}{\chi^3}\right)_{1 \times 2} \\ \frac{1}{\chi^3} & \left(\frac{1}{\chi^3}\right)_{1 \times 3} & \frac{1}{\chi^4} & \frac{1}{\chi^4} & \frac{\mu}{\chi^3} & \frac{1}{\chi^4} & \left(\frac{\mu}{\chi^3}\right)_{1 \times 2} \\ \mu & \left(\frac{\mu}{\chi^3}\right)_{1 \times 3} & \frac{1}{\chi^2} & \frac{\mu \mathcal{G}}{\chi^3} & \mu & \frac{\mu^2 \mathcal{G}}{\chi} & \left(\frac{\mu}{\chi}\right)_{1 \times 2} \\ \frac{\mu \mathcal{G}}{\chi} & \left(\frac{\mu \mathcal{G}}{\chi^4}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^3} & \frac{\mu \mathcal{G}^2}{\chi^4} & \frac{\mu^2 \mathcal{G}}{\chi} & \mu & \left(\frac{\mu \mathcal{G}}{\chi^2}\right)_{1 \times 2} \\ \frac{\chi}{\mu \chi^2} & \left(\frac{1}{\chi^4}\right)_{1 \times 3} & \frac{1}{\mu \chi^3} & \frac{\mathcal{G}}{\chi^4} & \frac{1}{\chi^2} & \frac{\mathcal{G}}{\chi^3} & \left(\frac{\mu \mathcal{G}}{\chi^3}\right)_{1 \times 2} \\ \frac{1}{\chi^2} & \left(\frac{1}{\chi^4}\right)_{1 \times 3} & \frac{\mathcal{G}}{\chi^4} & \frac{1}{\mu \chi^3} & \frac{\mu}{\chi^2} & \frac{\mu \mathcal{G}}{\chi^3} & \left(\frac{\mu \mathcal{G}}{\chi^3}\right)_{1 \times 2} \\ \frac{1}{\chi} & \left(\frac{\mu}{\chi^3}\right)_{1 \times 3} & \frac{\mu \mathcal{G}}{\chi^3} & \frac{\mu \mathcal{G}}{\chi^3} & \frac{\mu}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \left(\frac{1}{\chi}\right)_{1 \times 2} \\ \frac{1}{\chi} & \left(\frac{\mu}{\chi^3}\right)_{1 \times 3} & \frac{\mu \mathcal{G}}{\chi^3} & \frac{\mu \mathcal{G}}{\chi^3} & \frac{\mu}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \left(\frac{1}{\chi}\right)_{1 \times 2} \\ \frac{1}{\chi} & \left(\frac{\mu}{\chi^3}\right)_{1 \times 3} & \frac{\mu \mathcal{G}}{\chi^3} & \frac{\mu \mathcal{G}}{\chi^3} & \frac{\mu}{\chi} & \frac{\mu \mathcal{G}}{\chi^2} & \left(\frac{1}{\chi}\right)_{1 \times 2} \end{bmatrix}.$$

• In addition we have in the right case

$$\frac{\partial \mathcal{F}_4^R}{\partial \mathcal{V}_4^R} = -\frac{1}{\chi} \begin{bmatrix} \frac{\xi L^4 \text{sign}(v_{4,\parallel})}{(G^2 + L^2)(1 - \xi)^3} & \frac{\xi L^3}{(1 - \xi)^3} \\ -\xi L^5 & -\xi L^4 \text{sign}(v_{4,\parallel}) \end{bmatrix} + O\left(\frac{\mu}{\chi} + \frac{\mu}{|Q_4|^2}\right),$$

where $\xi = \frac{|x_4|}{\chi}$.

and in the left case

$$\frac{\partial \mathcal{F}_4^L}{\partial \mathcal{V}_4^L} = -\frac{1}{\chi} \begin{bmatrix} \frac{\xi L^2 \text{sign}(v_{4,\parallel})}{(1 - \xi)^3} & \frac{\xi L^3}{(1 - \xi)^3} \\ -\xi L & -\xi L^2 \text{sign}(v_{4,\parallel}) \end{bmatrix} + O\left(\frac{\mu}{\chi}\right), \text{ where } \xi = \frac{|x_4|}{\chi}.$$

Proof. • **A formal computation.**

Using our notation, the two matrices are $\frac{d\mathcal{F}}{d\mathcal{V}}$. They are the coefficient matrices of the variational equations $\frac{d}{d\ell_4} \delta \mathcal{V} = \frac{d\mathcal{F}}{d\mathcal{V}} \delta \mathcal{V}$. We split each of the two matrices into nine blocks corresponding to $\frac{\partial \mathcal{F}_i}{\partial \mathcal{V}_j}$, where $i, j = 3, 1, 4$.

Notice that $\mathcal{F} = \frac{dt}{d\ell_4} J \frac{dH}{d\mathcal{V}}$ where J is the standard symplectic matrix. Then we get the formal expression to calculate the two matrices:

$$(7.1) \quad \frac{d\mathcal{F}}{d\mathcal{V}} = J \frac{dH}{d\mathcal{V}} \otimes \frac{d}{d\mathcal{V}} \frac{dt}{d\ell_4} + \frac{dt}{d\ell_4} \frac{d}{d\mathcal{V}} \left(J \frac{dH}{d\mathcal{V}} \right).$$

Note that $\frac{d}{d\mathcal{V}} \frac{dt}{d\ell_4}$ is done in Lemma 7.1 and $J \frac{dH}{d\mathcal{V}} = \mathcal{F}$ is done in Corollary 6.2, the term $\frac{dt}{d\ell_4} = O(1)$ and the new term we need to consider is $\frac{d}{d\mathcal{V}} J \frac{dH}{d\mathcal{V}}$. For

$i, j = 3, 1, 4$, we have

$$\begin{aligned}
 \frac{d}{d\mathcal{V}_i} J_j \frac{\partial H}{\partial \mathcal{V}_j} &= \left(\frac{\partial}{\partial \mathcal{V}_i} + \frac{\partial L_4}{\partial \mathcal{V}_i} \frac{\partial}{\partial L_4} \right) J_j \frac{\partial H}{\partial \mathcal{V}_j} \\
 (7.2) \quad &= \frac{\partial \mathcal{X}_i}{\partial \mathcal{V}_i} J_j \frac{\partial^2 H}{\partial \mathcal{X}_i \partial \mathcal{X}_j} \frac{\partial \mathcal{X}_j}{\partial \mathcal{V}_j} + J_j \frac{\partial H}{\partial \mathcal{X}_j} \frac{\partial^2 \mathcal{X}_j}{\partial \mathcal{V}_i \partial \mathcal{V}_j} \\
 &\quad + \left(\frac{\partial \mathcal{X}_4}{\partial L_4} J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_j} \frac{\partial \mathcal{X}_j}{\partial \mathcal{V}_j} + J_j \frac{\partial H}{\partial \mathcal{X}_j} \frac{\partial^2 \mathcal{X}_j}{\partial L_4 \partial \mathcal{V}_j} \right) \otimes \frac{\partial L_4}{\partial \mathcal{V}_i},
 \end{aligned}$$

where J_i is the standard symplectic matrix in the i component. We know by Lemma 6.2 that $\frac{\partial \mathcal{X}_3}{\partial \mathcal{V}_3} = O(1)$, $\frac{\partial \mathcal{X}_1}{\partial \mathcal{V}_1} = \text{Id}_4$ and $\frac{\partial \mathcal{X}_4}{\partial L_4}, \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} = O(\ell_4)$ according to Lemma A.2.

Moreover, $\frac{\partial H}{\partial \mathcal{X}_j}, \frac{\partial^2 H}{\partial \mathcal{X}_i \partial \mathcal{X}_j}$ are done in Lemma 6.3, $\frac{\partial^2 \mathcal{X}_j}{\partial \mathcal{V}_i \partial \mathcal{V}_j}$ and $\frac{\partial^2 \mathcal{X}_j}{\partial L_4 \partial \mathcal{V}_j}$ are done in

Lemma A.4, and finally, $\frac{\partial L_4}{\partial \mathcal{V}_i}$ is done in Lemma 7.1. So in the following we analyze the two matrices blockwise.

$$\bullet \frac{\partial \mathcal{F}_3}{\partial \mathcal{V}_3}.$$

For this block the $(2, 1)$ entry is special. All the remaining entries are done together.

Using the Hamiltonian equations (6.3), we see the $(2, 1)$ entry is

$$\begin{aligned}
 &\frac{\partial}{\partial L_3} \left(\frac{dt}{d\ell_4} \left(\frac{m_3 k_3^2}{L_3^3} + \frac{\partial U}{\partial L_3} \right) \right) \\
 &= -\frac{\partial}{\partial L_3} \left(m_3 k_3^2 \left(1 + L_3^2 \left(\frac{v_1^2}{2m_1} - \frac{k_1}{|x_1|} \right) + 3UL_3^2 \right) - L_3^6 \frac{\partial U}{\partial L_4} + L_3^3 \frac{\partial U}{\partial L_3} + h.o.t. \right)
 \end{aligned}$$

The leading term is $\frac{\partial L_3^2}{\partial L_3} \cdot \left(\frac{v_1^2}{2m_1} - \frac{k_1}{|x_1|} \right) = O(\mu)$ since

$$m_1 = O(1/\mu), \quad L_3 = O(1), \quad v_1 = O(1), \quad |x_1| = O(\chi).$$

All the other terms involve U , which are at most $O\left(\frac{1}{\chi}\right)$ for left case and $O\left(\frac{\mu}{|\ell_4|^3 + 1}\right) + O\left(\frac{1}{\chi}\right)$ for the right case. This completes the estimate of the $(2, 1)$ entry.

For the first, third and fourth rows, we use formula (7.1). The first summand in (7.1) gives $(u(\ell_4), u(\ell_4), u(\ell_4)) \otimes (u(\ell_4), 1, u(\ell_4), u(\ell_4))$ for the three rows. The second summand is given by (7.2). The first and second term in (7.2) as well as the

summand in the third term $J_3 \frac{\partial H}{\partial \mathcal{X}_3} \frac{\partial^2 \mathcal{X}_3}{\partial L_4 \partial \mathcal{V}_3} \otimes \frac{\partial L_4}{\partial \mathcal{V}_3}$ in (7.2) has the same estimate

as $\frac{\partial U}{\partial \mathcal{X}_3}, \frac{\partial^2 U}{\partial \mathcal{X}_3^2} \lesssim u(\ell_4)$ in the right case and $\frac{1}{\chi^3}$ in the left case, as we get in Lemma

6.3, since $\frac{\partial \mathcal{X}_3}{\partial \mathcal{V}_3}, \frac{\partial^2 \mathcal{X}_3}{\partial \mathcal{V}_3^2}, \frac{\partial^2 \mathcal{X}_3}{\partial L_4 \partial \mathcal{V}_3} = O(1)$ and $\frac{\partial L_4}{\partial \mathcal{V}_3} = O(1)$ using Lemma 7.1. The

third term in (7.2) is estimated as $\frac{\mu}{|\ell_4|^3 + 1}$ in the right case and $\frac{\mu}{\chi^3}$ in the left case

using the estimate $\frac{\partial^2 U}{\partial x_3 \partial x_4}$ in Lemma 6.3 and the fact that $\frac{\partial \mathcal{X}_4}{\partial L_4} = O(\ell_4)$.

$$\bullet \frac{\partial \mathcal{F}_3}{\partial \mathcal{V}_1}.$$

The first summand in (7.1) gives $(u(\ell_4), 1, u(\ell_4), u(\ell_4)) \otimes (\frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi})$ in the right case with $u(\ell_4)$ replaced by $\frac{1}{\chi^3}$ in the left case. This gives all the entries of the right matrix and the second row and the third column of the left matrix. It remains to show the second summand in (7.1) is small. The first summand in (7.2) has the same estimate as $\frac{\partial^2 H}{\partial \mathcal{X}_1 \partial \mathcal{X}_3} \lesssim \frac{1}{\chi^4}$ given by Lemma 6.3. This gives all the remaining entries of the left matrix. The second summand in (7.2) vanishes since $\frac{\partial^2 \mathcal{X}_3}{\partial \mathcal{V}_1 \partial \mathcal{V}_3} = 0$. It remains to show that the third summand in (7.2) is small, we notice that $\frac{\partial L_4}{\partial \mathcal{V}_1} \lesssim \left(\frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi} \right)$ using Lemma 7.1. We also know $J_3 \frac{\partial H}{\partial \mathcal{X}_3} \frac{\partial^2 \mathcal{X}_3}{\partial L_4 \partial \mathcal{V}_3} \lesssim u(\ell_4)$ in the right case and $\lesssim 1/\chi^3$ in the left case done in the previous bullet point. We only need to show $\left(\frac{\partial \mathcal{X}_4}{\partial L_4} J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_3} \frac{\partial \mathcal{X}_3}{\partial \mathcal{V}_3} \right)$ is smaller than $u(\ell_4)$. Indeed, we know that $\frac{\partial \mathcal{X}_4}{\partial L_4} = O(\ell_4)$, and $\frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_3} \lesssim \frac{\mu}{\ell_4^4 + 1}$ in the right case and $\frac{\mu}{\chi^4}$ in the left case using Lemma 6.3. This concludes the estimates for the third summand.

• $\frac{\partial \mathcal{F}_3}{\partial \mathcal{V}_4}$.

The second row is handled in a similar manner to the $\frac{\partial \mathcal{F}_3}{\partial \mathcal{V}_3}$ block. Thus

$$\begin{aligned} \frac{\partial}{\partial \mathcal{V}_4} \left(\frac{dt}{d\ell_4} \left(\frac{m_3 k_3^2}{L_3^3} + \frac{\partial U}{\partial L_3} \right) \right) &= -\frac{\partial}{\partial \mathcal{V}_4} \left(m_3 k_3^2 \left(1 + L_3^2 \left(\frac{v_1^2}{2m_1} - \frac{k_1}{|x_1|} \right) + 3UL_3^2 \right) \right. \\ &\quad \left. - L_3^6 \frac{\partial U}{\partial L_4} + L_3^3 \frac{\partial U}{\partial L_3} + h.o.t. \right) = -\frac{\partial}{\partial \mathcal{V}_4} \left(3UL_3^2 - L_3^6 \frac{\partial U}{\partial L_4} + L_3^3 \frac{\partial U}{\partial L_3} + h.o.t. \right) \end{aligned}$$

The leading term is given by $\frac{\partial}{\partial \mathcal{V}_4} \frac{\partial U}{\partial L_4} = \left(\frac{\partial \mathcal{X}_4}{\partial L_4} J_j \frac{\partial^2 U}{\partial x_4^2} \frac{\partial x_4}{\partial \mathcal{V}_4} \right) \lesssim w(\ell_4)$ in the right case and $\frac{1}{\chi}$ in the left case using Lemma 6.3.

Next, the first summand in (7.1) gives $(u(\ell_4), u(\ell_4), u(\ell_4)) \otimes (w(\ell_4), w(\ell_4))$ in the right case with $u(\ell_4), w(\ell_4)$ replaced by $\frac{1}{\chi^3}, \frac{1}{\chi}$ respectively in the left case. This is smaller than what we have stated in the lemma. It remains to consider (7.2). The first summand in (7.2) has the estimate as $\frac{\mu}{\ell_4^3 + 1}$ in the right case and $\frac{\mu}{\chi^3}$ in the left case since we have the estimate of $\frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_3} \lesssim \frac{\mu}{\ell_4^4 + 1}$ in the right case and $\frac{\mu}{\chi^4}$ in the left case in Lemma 6.3. This is what we have stated in the lemma. The second summand is $u(\ell_4)v(\ell_4)$ in the right case and $\mu\mathcal{G}/\chi^5$ in the left case using Lemma 6.3 for $\frac{\partial H}{\partial \mathcal{X}_3}$ and Lemma 7.1 for $\frac{\partial^2 \mathcal{X}_3}{\partial \mathcal{V}_4 \partial \mathcal{V}_3}$. The same estimate holds for the $J_3 \frac{\partial H}{\partial \mathcal{X}_3} \frac{\partial^2 \mathcal{X}_3}{\partial L_4 \partial \mathcal{V}_3} \otimes \frac{\partial L_4}{\partial \mathcal{V}_4}$ in the third summand. This two estimates is much smaller than the estimate stated in the lemma. For the remaining part in the third

summand in (7.2), we estimate $\frac{\partial L_4}{\partial \mathcal{V}_4}$ as $(v(\ell_4), v(\ell_4))$ in the right case and $(\frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2})$ in the left case using Lemma 7.1. Next $\left(\frac{\partial \mathcal{X}_4}{\partial L_4} J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_3} \frac{\partial \mathcal{X}_3}{\partial \mathcal{V}_3} \right) \lesssim \frac{\mu}{|\ell_4|^3 + 1}$ in the right case and $\frac{\mu}{\chi^3}$ in the left case using Lemma 6.3 as it was done in the estimate of $\frac{\partial \mathcal{F}_3}{\partial \mathcal{V}_3}$. So the third summand is also smaller than what is stated in the lemma.

For the next three blocks $\frac{\partial \mathcal{F}_1}{\partial \mathcal{V}_i}$, $i = 3, 1, 4$, the second summand in (7.2) vanishes since $\frac{\partial^2 \mathcal{X}_1}{\partial \mathcal{V}_i \partial \mathcal{V}_1} = \frac{\partial \text{Id}}{\partial \mathcal{V}_i} = 0$ so does $J_1 \frac{\partial H}{\partial \mathcal{X}_1} \frac{\partial^2 \mathcal{X}_1}{\partial L_4 \partial \mathcal{V}_1} \otimes \frac{\partial L_4}{\partial \mathcal{V}_i}$, $i = 3, 1, 4$ in the third summand. We do not consider them in the three blocks.

$$\bullet \frac{\partial \mathcal{F}_1}{\partial \mathcal{V}_3}.$$

The first summand in (7.1) gives $\left(\mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3} \right) \otimes (1, u(\ell_4), u(\ell_4), u(\ell_4))$ in the right case with $u(\ell_4)$ replaced by $\frac{1}{\chi^3}$ in the left case. The first summand in (7.2) has the same estimate as $\frac{\partial^2 H}{\partial \mathcal{X}_3 \partial x_1} \lesssim \frac{1}{\chi^4}$, $\frac{\partial^2 H}{\partial \mathcal{X}_3 \partial v_1} = 0$ using Lemma 6.3, whose contribution to the current block is $\begin{bmatrix} 0_{2 \times 4} \\ (1/\chi^4)_{2 \times 4} \end{bmatrix}$.

Finally, we consider the third summand in (7.2). $\frac{\partial L_4}{\partial \mathcal{V}_3} \lesssim (1, u(\ell_4), u(\ell_4), u(\ell_4))$ in the right case with $u(\ell_4)$ replaced by $\frac{1}{\chi^3}$ in the left case using Lemma 7.1. Next we consider $\left(\frac{\partial \mathcal{X}_4}{\partial L_4} J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} \frac{\partial \mathcal{X}_1}{\partial \mathcal{V}_1} \right)$. This is a vector of four entries whose first two entries are $\frac{\partial^2 H}{\partial L_4 \partial v_1} = \frac{\partial v_1}{\partial L_4} = 0$ and whose last two entries are bounded by $\frac{1}{\chi^2}$, since we have $\frac{\partial \mathcal{X}_4}{\partial L_4} = O(\ell_4)$, and $\frac{\partial^2 H}{\partial \mathcal{X}_4 \partial x_1}$ is bounded by $\frac{1}{\chi^3}$ using Lemma 6.3. This implies we need to compare $\left(\frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3} \right) \otimes (1, (u(\ell_4))_{1 \times 3}, 1/\chi^4)$ and $\frac{1}{\chi^2} (1, (u(\ell_4))_{1 \times 3})$ in the right case and with u replaced by $1/\chi^3$ in the left case, to get a larger one as the last row of the right matrix.

$$\bullet \frac{\partial \mathcal{F}_1}{\partial \mathcal{V}_1}.$$

The first summand in (7.1) gives $\left(\mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3} \right) \otimes \left(\frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi} \right)$. The first summand in (7.2) is $\begin{bmatrix} \frac{\partial^2 H}{\partial x_1 \partial v_1} & \frac{\partial^2 H}{\partial v_1 \partial v_1} \\ -\frac{\partial^2 H}{\partial x_1^2} & -\frac{\partial^2 H}{\partial x_1 \partial v_1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m_1} \text{Id} \\ \left(\frac{3k_1 x_1 \otimes x_1}{|x_1|^5} \right) - \frac{k_1 \text{Id}}{|x_1|^3} & 0 \end{bmatrix}$, where $\frac{\partial^2 H}{\partial x_1^2}$ is given by $\frac{\partial^2}{\partial x_1^2} \frac{k_1}{|x_1|}$. We compare the two matrices using $x_1 = O(\chi, \mu \mathcal{G})$ to get the first three rows and the (8, 5), (8, 6)th entry in both matrices. Finally we

consider the third summand in (7.2). We notice that $\frac{\partial L_4}{\partial \mathcal{V}_1} \lesssim \left(\frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi} \right)$.

The term $\left(\frac{\partial \mathcal{X}_4}{\partial L_4} J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} \frac{\partial \mathcal{X}_1}{\partial \mathcal{V}_1} \right) \lesssim \left(0, 0, \frac{1}{\chi^2}, \frac{1}{\chi^2} \right)$ as in the previous paragraph. This gives us the remaining two entries (8, 7), (8, 8) in the fourth row of the two matrices because of $1/\chi^2 \gg \mathcal{G}/\chi^3$.

$$\bullet \frac{\partial \mathcal{F}_1}{\partial \mathcal{V}_4}.$$

The first summand in (7.1) gives $\left(\mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3} \right) \otimes (w(\ell_4), w(\ell_4))$ in the right case with $w(\ell_4)$ replaced by $\frac{1}{\chi}$ in the left case using Lemma 7.1 and Corollary 6.2. This gives the first two rows in both matrices. The two remaining summands in (7.2) can be written together as $\left(J_1 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} \right) \left(\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} + \frac{\partial \mathcal{X}_4}{\partial L_4} \otimes \frac{\partial L_4}{\partial \mathcal{V}_4} \right)$. The first two rows here are zero because $\frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{V}_1} = 0$. The nonzero entries of $\left(J_1 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} \right) \frac{\partial \mathcal{X}_4}{\partial L_4}$ are of order $1/\chi^2$ as in the previous paragraph and $\frac{\partial L_4}{\partial \mathcal{V}_4}$ is estimated as $(v(\ell_4), v(\ell_4))$ in the right case and $(\mu \mathcal{G}/\chi^2, \mu \mathcal{G}/\chi^2)$ in the left case using Lemma 7.1, so the tensor part is $O(v(\ell_4)/\chi^2)$ in the left case and $O(\mu \mathcal{G}/\chi^4)$ in the left case. Moreover, for the remaining part we already have $J_1 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} = O(1/\chi^3)$ using Lemma 6.3.

Though $\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ can be as large as $O(\chi)$, using the same argument as the proof of Lemma 6.4 and Corollary 6.2 (the almost orthogonality of $\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ with x_4 and x_1), we find $\left(J_1 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} \right) \left(\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} \right) \lesssim v(\ell_4)/\chi$ in the right case and $O(\mu \mathcal{G}/\chi^3)$ in the left case. In the left case, we see that this $O(\mu \mathcal{G}/\chi^3)$ gives the last two rows in the current block since it is larger than other contributions. In the right case, we need to compare $\frac{\mathcal{G}}{\chi^3} w(\ell_4)$, $1/\chi^2 v(\ell_4)$, and $\frac{1}{\chi} v(\ell_4)$. It turns out that $v(\ell_4)/\chi$ dominates.

$$\bullet \frac{\partial \mathcal{F}_4}{\partial \mathcal{V}_3}.$$

The first summand in (7.1) gives $(v(\ell_4), v(\ell_4)) \otimes (1, u(\ell_4), u(\ell_4), u(\ell_4))$ in the right case, with $u(\ell_4), v(\ell_4)$ replaced by $\frac{1}{\chi^3}, \frac{\mu \mathcal{G}}{\chi^2}$ respectively in the left case. This does not show up in the statement of the lemma. The first summand in (7.2) $\frac{\partial \mathcal{X}_3}{\partial \mathcal{V}_3} \left(J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_3} \right) \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ is estimated as $\frac{\mu}{|\ell_4|^3 + 1}$ in the right case and $\frac{\mu}{\chi^3}$ in the left case using Lemma 6.3. This gives the second, third and fourth columns of the block.

Next, consider the second summand in (7.2). We have $\frac{\partial H}{\partial \mathcal{X}_4} \lesssim \frac{1}{\chi^2} + \frac{\mu}{\ell_4^4 + 1}$ in the right case and $\lesssim \frac{1}{\chi^2}$ in the left case using Lemma 6.3, and $\frac{\partial^2 \mathcal{X}_4}{\partial \mathcal{V}_3 \partial \mathcal{V}_4} =$

$\frac{\partial^2 \mathcal{X}_4}{\partial \mathcal{L}_4 \partial \mathcal{V}_4} \frac{\partial L_4}{\partial \mathcal{V}_3} \lesssim \ell_4(1, (u(\ell_4))_{1 \times 3})$ in the right case and $\lesssim \left(\chi, \left(\frac{1}{\chi^2} \right)_{1 \times 3} \right)$ in the left case using Lemma 7.1 and A.4. This gives the first column as $w(\ell_4)$ in the right case and $1/\chi$ in the left case.

It remains to consider the third summand in (7.2). We have $\frac{\partial L_4}{\partial \mathcal{V}_3} \lesssim (1, (u(\ell_4))_{1 \times 3})$ in the right case with $u(\ell_4)$ replaced by $\frac{1}{\chi^3}$ in the left case. Next $J_4 \frac{\partial H}{\partial \mathcal{X}_4} \frac{\partial \mathcal{X}_4}{\partial L_4 \partial \mathcal{V}_4}$ is bounded by $\ell_4 \left(\frac{1}{\chi^2} + \frac{\mu}{\ell_4^4 + 1} \right) \lesssim w(\ell_4)$ in the right case and $\lesssim \frac{1}{\chi}$ in the left case using Lemma 6.3 and A.4. Next, $\frac{\partial \mathcal{X}_4}{\partial L_4} \left(J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_4} \right) \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ is bounded by $v(\ell_4)$ in the right case and $\frac{\mu \mathcal{G}}{\chi^2}$ in the left case using Lemma 6.3 and the almost orthogonality of $\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ with x_4 and x_1 . So the third summand in (7.2) has the estimates $(w(\ell_4))_{1 \times 2} \otimes (1, (u(\ell_4))_{1 \times 3})$ in the right case and $\left(\frac{1}{\chi} \right)_{1 \times 2} \otimes \left(1, \left(\frac{1}{\chi} \right)_{1 \times 3} \right)$. So the third summand does not have new contribution to the block.

$$\bullet \frac{\partial \mathcal{F}_4}{\partial \mathcal{V}_1}.$$

The first summand in (7.1) gives $(v(\ell_4), v(\ell_4)) \otimes \left(\frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi} \right)$ in the right case and with $v(\ell_4)$ replaced by $\frac{\mu \mathcal{G}}{\chi^2}$ in the left case. This estimate does not show up in the statement of the lemma. The first summand in (7.2) $\left(J_4 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_1} \right) \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ is estimated by $\frac{\mu \mathcal{G}}{\chi^3}$ in the left case and $v(\ell_4)/\chi$ in the right case using Lemma 6.3 and the almost orthogonality of $\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}$ with x_4 and x_1 (see also the above argument for $\frac{\partial \mathcal{F}_4}{\partial \mathcal{V}_1}$ block). This gives the first and second columns of the current block. The

second summand in (7.2) vanishes since $\frac{\partial^2 \mathcal{X}_4}{\partial \mathcal{V}_1 \partial \mathcal{V}_4} = 0$. Finally, we consider the third summand in (7.2). We have $\frac{\partial L_4}{\partial \mathcal{V}_1} \lesssim \left(\frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi} \right)$. The estimate of $\frac{\partial \mathcal{X}_4}{\partial L_4} \left(J_4 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_4} \right) \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} + J_4 \frac{\partial H}{\partial \mathcal{X}_4} \frac{\partial \mathcal{X}_4}{\partial L_4 \partial \mathcal{V}_4}$ was done in the previous paragraph. We find the third summand in (7.2) contributes to the third and fourth columns of the current block.

$$\bullet \frac{\partial \mathcal{F}_4}{\partial \mathcal{V}_4}.$$

The leading contribution is given by the first and second summands in (7.2),

$$\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} J_4 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_4} \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} + J_4 \frac{\partial H}{\partial \mathcal{X}_4} \frac{\partial^2 \mathcal{X}_4}{\partial \mathcal{V}_4 \partial \mathcal{V}_4} = O\left(\frac{1}{\chi}\right)$$

since we have $\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4}, \frac{\partial^2 \mathcal{X}_4}{\partial \mathcal{V}_4 \partial \mathcal{V}_4} = O(\chi)$ using Lemma A.2 and A.4, and $\frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_4} = O(1/\chi^3)$, $\frac{\partial H}{\partial \mathcal{X}_4} = O(1/\chi^2)$. We next show the other summands are small. The first summand in (7.1) gives $(v(\ell_4), v(\ell_4)) \otimes (w(\ell_4), w(\ell_4))$ in the right case with $v(\ell_4), w(\ell_4)$ replaced by $\frac{\mu \mathcal{G}}{\chi^2}, \frac{1}{\chi}$ respectively in the left case. Then we consider the second summand in (7.2). We have $\frac{\partial L_4}{\partial \mathcal{V}_4} \lesssim \left(\frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2} \right)$. Next, $\left(J_4 \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_4} \right) \frac{\partial \mathcal{X}_4}{\partial L_4} + \left(J_4 \frac{\partial H}{\partial \mathcal{X}_4} \right) \frac{\partial^2 \mathcal{X}_4}{\partial L_4 \partial \mathcal{V}_4}$ is estimated as before. Both contributions are much smaller than $1/\chi$, so they do not enter the estimates of the matrices.

Part (a) and (b) are complete now. Finally, we show part (c).

According to the last bullet point, the leading terms in $\frac{\partial \mathcal{F}_4}{\partial \mathcal{V}_4}$ come from

$$\frac{dt}{d\ell_4} \left(\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} \left(J_j \frac{\partial^2 H}{\partial \mathcal{X}_4 \partial \mathcal{X}_4} \right) \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}_4} + J_4 \frac{\partial H}{\partial \mathcal{X}_4} \frac{\partial^2 \mathcal{X}_4}{\partial \mathcal{V}_4 \partial \mathcal{V}_4} \right) = L_4^3 \left(\frac{\partial}{\partial G_4}, \frac{\partial}{\partial g_4} \right) \left[\begin{array}{c} \frac{\partial x_4}{\partial g_4} \cdot \frac{\partial U}{\partial x_4} \\ - \frac{\partial x_4}{\partial G_4} \cdot \frac{\partial U}{\partial x_4} \end{array} \right].$$

Let us now look at U^R in (4.3). Only those terms in U^R containing both x_4 and x_1 can be as large as $O(1/\chi)$ according to item (1) of Lemma 6.1. So we only need to consider the following three terms in U^R ,

$$- \left(\frac{1}{\mu \left| x_1 + \frac{\mu}{1+2\mu} x_4 + \frac{\mu}{1+\mu} x_3 \right|} + \frac{1}{\left| x_1 + \frac{\mu}{1+2\mu} x_4 - \frac{1}{1+\mu} x_3 \right|} + \frac{1}{\left| x_1 - \frac{1+\mu}{1+2\mu} x_4 \right|} \right).$$

when we take two derivatives with respect to x_4 , a μ^2 factor will be multiplied to the first two terms, so that the first two terms would be $O(\mu)$ compared to the third term. So the leading contribution to $\frac{\partial^2 U^R}{\partial x_4^2}$ is given by $\frac{\partial^2}{\partial x_4^2} \frac{-1}{\left| x_1 - \frac{1+\mu}{1+2\mu} x_4 \right|}$.

The same analysis for U^L in (4.5) shows the leading contribution to $\frac{\partial^2 U^L}{\partial x_4^2}$ is given

$$\text{by } \frac{\partial^2}{\partial x_4^2} \frac{-1}{\left| x_1 + \frac{\mu}{1+\mu} + \frac{1+\mu}{1+2\mu} x_4 \right|}.$$

Consider the (9, 9) entry. The main contribution to this entry comes from

$$(7.3) \quad L_4^3 \frac{\partial}{\partial G_4} \left(\frac{\partial x_4}{\partial g_4} \cdot \frac{\partial U}{\partial x_4} \right) = (1 + O(\mu)) L_4^3 \frac{\partial}{\partial G_4} \left(\frac{\frac{\partial x_4}{\partial g_4} \cdot (x_4 - (1 + O(\mu)x_1))}{|x_4 - (1 + O(\mu)x_1)|^3} \right).$$

The numerator in the RHS equals to

$$(1 + O(\mu)) L_4^3 \frac{\partial}{\partial G_4} \left(\frac{\partial x_4}{\partial g_4} \cdot x_4 \right) - (1 + O(\mu)) L_4^3 \frac{\partial^2 x_4}{\partial G_4 \partial g_4} \cdot x_1.$$

The first term is $O(\chi)$ due to Lemma A.2(c) so the main contribution comes from the second term which is $O(\chi^2)$ using Lemma A.4. We use the same argument to

other entries to get

$$(7.4) \quad \frac{\partial \mathcal{F}_4^R}{\partial \mathcal{V}_4} = -L_3^3 \left[\begin{array}{cc} \frac{\partial^2 x_4}{\partial G \partial g} \cdot \frac{x_1}{|x_4 - x_1|^3} & \frac{\partial^2 x_4}{\partial g^2} \cdot \frac{x_1}{|x_4 - x_1|^3} \\ -\frac{\partial^2 x_4}{\partial G^2} \cdot \frac{x_1}{|x_4 - x_1|^3} & -\frac{\partial^2 x_4}{\partial G \partial g} \cdot \frac{x_1}{|x_4 - x_1|^3} \end{array} \right] + O\left(\frac{\mu}{\chi} + \frac{\mu}{|x_4|^3}\right).$$

Using Lemma A.4 we see that (9, 9) entry equals to

$$-\frac{L_4^5}{\sqrt{L_4^2 + G_4^2}} \frac{\chi \sinh u}{|x_4 - x_1|^3} + O\left(\frac{\mu}{\chi} + \frac{\mu}{|x_4|^3}\right).$$

Recall that $L_3 = L_4(1 + o(1))$ (due to (6.1)) and $\sinh u = \text{sign}(u) \frac{|\ell_4| L_4}{\sqrt{L_4^2 + G_4^2}}$ (due to (A.4)). Since Lemma 6.2 implies that $|x_4| = |\ell_4|/L_4^2(1 + o(1))$ we obtain that $O(1/\chi)$ -term in (9, 9) is asymptotic to

$$-\frac{L^4 \text{sign}(u)}{L^2 + G^2} \frac{\chi |x_4|}{(\chi - |x_4|)^3}.$$

Since u and $v_{4,\parallel}$ have opposite signs we obtain the asymptotics of $O(1/\chi)$ -term claimed in part (c) of the Lemma 7.2 for the (9, 9) entry. The analysis of other entries of $\frac{\partial \mathcal{F}_4^R}{\partial \mathcal{V}_4}$ is similar.

Next, we consider the left case. The argument is the same except the following differences. First the error term in (7.3) is now $O(\mu/\chi)$ since $\mu/|x_4|^3$ should be replaced by $1/\chi^3$ as usual. Next U^L is roughly $\frac{1}{|x_4 + x_1|}$ up to some μ error, which differs from U^R by a “-” sign. Then we have now the asymptotic expression of (7.4) follows directly from Lemma A.4(c). \square

7.2. Estimates of the solution of the variational equations.

Estimates of matrices N_1 , M , N_5 and the $(I)_{44}$, $(III)_{44}$, $(V)_{44}$ blocks in Proposition 5.1. From one Poincaré section to the next, it takes time of order $O(\chi)$.

Step 1, the matrix M .

Let us first explain how to get the matrix M . Since the right matrix of Lemma 7.2 has constant entries, which we denote by K temporarily, M can be estimated by the fundamental solution of the ODE $X' = K \cdot X$, that is, by $X(\chi) = e^{K\chi} = \sum_{n=0}^{\infty} \frac{1}{n!} (K\chi)^n$. Note that K has positive entries. We claim that in fact

$$(7.5) \quad e^{K\chi} - \text{Id}_{10} = O(K\chi + \frac{1}{2}(K\chi)^2).$$

Indeed a brute force force calculation (which can be done on the computer) shows that $K\chi^3 \leq C_3(K\chi + (K\chi)^2)$. This allows to get inductively that

$$(7.6) \quad K^n \leq C_n(K\chi + (K\chi)^2) \text{ where } C_n = C_3(1 + C_3)^{n-3}.$$

Summing the series for $e^{K\chi}$ we obtain (7.5).

Let us discuss briefly how we use computer. Each entry of the product $(\chi K)(\chi K)$ has ten summands. However, we are only interested the leading term for large χ and small μ since in the statement of Proposition 5.1, we only need to bound N_1, N_5, M by some matrices in the sense of \lesssim . So we write simple codes to compare the ten summands and pick out the summand giving the largest value. Namely consider the product $C = AB$, instead of using $\sum_k A_{ik} B_{kj}$ as C_{ij} , we use $\max_k A_{ik} B_{kj}$.

Step 2, the matrices N_1, N_5 .

Denote the ODE by $\frac{dY}{dt} = \Lambda(t)Y$ with the initial condition $Y(0) = \text{Id}_{10}$. Using the Picard iteration, the solution is

$$(7.7) \quad \begin{aligned} Y(t) &= \text{Id} + \int_0^t \Lambda \cdot Y(s) ds = \text{Id} + \int_0^t \Lambda dt + \int_0^t \Lambda \left(\int_0^s \Lambda(\tau) d\tau \right) ds + \cdots \\ &:= \text{Id} + I_1(t) + I_2(t) + \cdots \end{aligned}$$

where I_i is the i -th iterated integral. We will show that for some $k \geq 3$ independent of μ, χ ,

$$(7.8) \quad I_k(t) \leq \bar{\gamma}(I_1(t) + I_2(t))$$

for all $t \in [0, \chi/2]$ and some $\bar{\gamma} \in (0, 1/2)$. Inductively we have

$$I_n(t) \leq \bar{\gamma}(I_{n-k+1}(t) + I_{n-k+2}(t)), \quad \forall n \geq k.$$

This gives

$$\begin{aligned} Y(t) &\leq \text{Id} + I_1(t) + \cdots + I_{k-1}(t) + \sum_{i=1}^{\infty} \bar{\gamma}(I_i(t) + I_{i+1}(t)) \\ &= \text{Id} + I_1(t) + \cdots + I_{k-1}(t) + 2\bar{\gamma}Y(t) - 2\bar{\gamma}\text{Id} - \bar{\gamma}I_1(t). \end{aligned}$$

Therefore

$$(7.9) \quad Y(t) \leq \text{Id} + \frac{1}{1-2\bar{\gamma}}((1-\bar{\gamma})I_1(t) + I_2(t) + \cdots + I_{k-1}(t)).$$

We will also show that

$$(7.10) \quad I_3(t) \leq \gamma_3(I_1(t) + I_2(t))$$

for some large constant γ_3 . (7.10) implies that

$$(7.11) \quad I_j(t) \leq \gamma_j(I_1(t) + I_2(t))$$

for all $j \leq k$. (7.11) together with (7.9) gives

$$(7.12) \quad Y(t) \leq \text{Id} + C(I_1(t) + I_2(t))$$

for some constant C . Using computer similarly to M -estimate we see that (7.12) implies our estimates for N_1 and N_5 .

It remains to establish (7.8) and (7.10). We begin with (7.10) which is checked by computer. Then we (7.10) and the fact that the domain of integration in (7.7) shrinks as $1/k!$ to deduce (7.8) from (7.10).

We write Mathematica codes as before to compute the matrix products. The following observations which allow us to reduce (7.8) to computing products of constant matrices, simplify the calculation significantly. In u, v, w , we replace $\frac{\mu}{\ell_4^3 + 1}$ by $\frac{\mu}{|\ell_4|^3}$ with ℓ_4 lying between 1 and $O(\chi)$. Recall that $\frac{\mu}{|\ell_4|^3}$ is the correct bound of terms of

the form $\frac{\mu|x_3|}{|x_4|^3}$ in Lemma 6.2, while $\frac{\mu}{\ell_4^3+1}$ was used to show that the denominator cannot be zero.

For N_1 , we pick a small constant ϵ_0 which is independent of μ, χ , so that $\int_{\ell_4^i}^{\ell_4^i+\epsilon_0} \frac{\mu}{\ell_4^3} d\ell_4 = \frac{\mu\epsilon_0}{(\ell_4^i)^3} + O(\epsilon_0^2)$ where $\ell_4^i = O(1) \neq 0$ is the initial ℓ_4 . Inequality (7.8) holds for $\ell_4 \in [\ell_4^i, \ell_4^i + \epsilon_0]$ for ϵ_0 small enough. For $\ell_4 \geq \ell_4^i + \epsilon_0$, we have $\int_{\ell_4^i}^{\ell_4} \frac{\mu}{s^3} ds = \frac{\mu}{2(\ell_4^i)^2} - \frac{\mu}{2(\ell_4)^2} = O(\mu)$, as $\ell_4 \rightarrow \infty$, $\mu \rightarrow 0$. So we replace all the integrals $\int_{\ell_4^i}^{\ell_4} \frac{\mu}{s^3} ds$ by μ in the sense of “ \sim ”. After integration in (7.8), there is no terms of the form $1/\ell_4^k$, $k > 0$.

Notice that we can decompose right matrix in Lemma 7.2 as $K + \frac{\mu}{\ell_4^3}B$, where K and B do not depend on ℓ , and K is exactly the same as in (7.5). We have

$$\begin{aligned} I_1 &\sim \ell_4 K + \mu B, \quad I_2 \sim \ell_4^2 K^2 + \mu \ell_4 K B + \mu B K + \mu^2 B^2, \\ I_3 &\sim \ell_4^3 K^3 + \mu \ell_4^2 K^2 B + \mu \ell_4 K B K + \mu^2 \ell_4 K B^2 + \mu \ln \ell_4 B K^2 + \mu^2 (B K B + B^2 K) + \mu^3 B^3. \end{aligned}$$

We separate terms in $I_1 + I_2$ into two groups:

$$\ell_4 K + \ell_4^2 K^2 + \mu \ell_4 K B, \text{ and } \mu(B + B K) + \mu^2 B.$$

Correspondingly we separate terms in I_3 into two groups:

$$\ell_4^3 K^3 + \mu \ell_4^2 K^2 B + \mu \ell_4 K B K + \mu^2 \ell_4 K B^2 + \mu \ln \ell_4 B K^2, \text{ and } \mu^2 (B K B + B^2 K) + \mu^3 B^3.$$

We expect the first (resp. second) group in $I_1 + I_2$ bounds the first (resp. second) group in I_3 . This is true for most of the entries with a few exceptions bounded with the help of the other group. Here we use computer to check $I_3 \leq C(I_1 + I_2)$ where the constant can be chosen and fixed, for instance 10.

For N_5 , we integrate ℓ_4 from $O(\chi)$ to $O(1)$, we use only $1/\chi$ in $w(\ell_4)$ when doing integration since its integral dominates the other term. Again we can decompose the right matrix in Lemma 7.2 as $K + \frac{\mu}{\ell_4^3}B$, whose integration for ℓ_4 from $\chi/2$ to 1 is $\sim (\chi/2 - \ell_4)K + \frac{\mu}{\ell_4^2}B$.

$$\begin{aligned} I_1 &= (\chi - \ell_4)K + \frac{\mu}{\ell_4^2}B, \quad I_2 = (\chi - \ell_4)^2 K^2 + \frac{\mu}{\ell_4} K B + \mu \frac{(\chi - \ell_4)}{\ell_4^2} B K + \frac{\mu^2}{\ell_4^4} B^2, \\ I_3 &\lesssim (\chi - \ell_4)^3 K^3 + \mu \ln \frac{\ell_4}{\chi} K^2 B + \mu \frac{\chi - \ell_4}{\ell_4} K B K + \frac{\mu^2}{\ell_4^3} K B^2 + \mu \frac{\chi - \ell_4}{\ell_4^2} B K^2 \\ &\quad + \frac{\mu^2}{\ell_4^3} B K B + \mu^2 \frac{\chi - \ell_4}{\ell_4^4} B^2 K + \frac{\mu^3}{\ell_4^6} B^3 \end{aligned}$$

Again we separate $I_1 + I_2$ into two groups:

$$(\chi - \ell_4)K + (\chi - \ell_4)^2 K^2 + \mu \frac{(\chi - \ell_4)}{\ell_4^2} B K, \text{ and } \frac{\mu}{\ell_4^2} B + \frac{\mu}{\ell_4} K B + \frac{\mu^2}{\ell_4^4} B^2.$$

We also separate I_3 into two groups:

$$(\chi - \ell_4)^3 K^3 + \mu \ln \frac{\ell_4}{\chi} K^2 B + \mu \frac{\chi - \ell_4}{\ell_4} K B K + \mu \frac{\chi - \ell_4}{\ell_4^2} B K^2 + \mu^2 \frac{\chi - \ell_4}{\ell_4^4} B^2 K,$$

and $\frac{\mu^2}{\ell_4^3}KB^2 + \frac{\mu^2}{\ell_4^3}BKB + \frac{\mu^3}{\ell_4^6}B^3$. We expect the first (resp. second) group in $I_1 + I_2$ bounds the first (resp. second) group in I_3 . This is true for most of the entries with a few exceptions bounded with the help of the other group. The fact that $\frac{1}{|\ell_4|^{k+1}} \lesssim \frac{1}{|\ell_4|^k}$ is used. Here again we use computer to check the inequality $I_3 \leq C(I_1 + I_2)$.

This checks (7.10). To prove (7.8) we consider all the possible products

$$(7.13) \quad K^{k_1}B^{l_1} \dots K^{k_m}B^{l_m}, \quad \sum_{i=1}^m k_i + l_i = k, \quad k_i \geq 0, \quad l_i \geq 0.$$

We consider N_1 , the analysis of N_5 is similar. We split $I_k = I_{k,0} + I_{k,1} + I_{k,2} + I_{k,3}$ where $I_{k,0}$ (respectively $I_{k,1}$, $I_{k,2}$) collect the terms where B appears 0 (respectively 1 or 2) times and $I_{k,3}$ represents the contribution of terms having 3 or more B s. We will show that for each ε and each $r \in \{0, 1, 2, 3\}$ we have

$$(7.14) \quad I_{k,r}(t) \leq \varepsilon(I_1(t) + I_2(t))$$

provided that k is large enough and $\mu \leq \mu_0$, $\chi \geq \chi_0$.

We first dispose of $I_{k,3}$. We already know that

$$(7.15) \quad I_{k,3}(t) \leq \gamma_k(I_1(t) + I_2(t)).$$

On the other hand all terms in $I_{k,3}$ have μ in at least 3d power while $I_1 + I_2$ has main terms of order at most μ^2 . So by decreasing μ we can improve (7.15) to (7.14).

$I_{k,0}$, $I_{k,1}$ and $I_{k,2}$ are treated similarly. Let us consider $I_{k,1}$ as an example. We have

$$I_{k,1} = \sum_p \int \dots \int K^{p-1}BK^{k-p} \frac{d\ell_1 \dots d\ell_k}{\ell_p^3}$$

where the integration is over $\ell \geq \ell_1 \geq \ell_2 \dots \geq \ell_k$. Using (7.6) we get

$$\begin{aligned} & \int \dots \int K^{p-1}BK^{k-p} \frac{d\ell_1 \dots d\ell_k}{\ell_p^3} = \\ & \iiint \left[(\chi K)^{p-1} \frac{(\ell - \ell_{p-1})^{p-1}}{\chi^{p-1}(p-1)!} \right] B \left[\frac{\ell_{p+1}^{k-p}}{(k-p)!} K^{k-p} \right] \frac{d\ell_{p-1}d\ell_p d\ell_{p+1}}{\ell_p^3} \leq \frac{C^k}{(p-1)!(k-p)!} \\ & \cdot \iiint ((\ell - \ell_{p-1})K + ((\ell - \ell_{p-1})\chi K)^2) B((\ell - \ell_{p-1})K + ((\ell - \ell_{p-1})K)^2) \frac{d\ell_{p-1}d\ell_p d\ell_{p+1}}{\ell_p^3} \\ & \leq \frac{C^k}{(\frac{k-1}{2})!} (I_3 + I_4 + I_5) \leq \frac{C^k[\gamma_3 + \gamma_4 + \gamma_5]}{(\frac{k-1}{2})!} (I_1 + I_2). \end{aligned}$$

where the first inequality uses that either $p-1$ or $k-p$ is at least $\frac{k-1}{2}$ and the second follows from (7.11). Summing over p we obtain

$$I_k \leq \frac{C^k k[\gamma_3 + \gamma_4 + \gamma_5]}{(\frac{k-1}{2})!} (I_1 + I_2).$$

$I_{k,0}$ and $I_{k,2}$ are treated in the same way showing that

$$I_{k,0} + I_{k,1} + I_{k,2} \leq \frac{\bar{C}C^k k}{(\frac{k-2}{3})!} [I_1 + I_2].$$

Now (7.8) follows from (7.10).

Step 3, the asymptotics of the $(N_1)_{44}, (M)_{44}, (N_5)_{44}$ blocks in Proposition 5.1 (b).

We have

$$\frac{d}{d\ell_4}\delta\mathcal{V}_4 = \sum_{i=3,1,4} \frac{\partial\mathcal{F}_4}{\partial\mathcal{V}_i} \cdot \delta\mathcal{V}_i, \quad \frac{\partial\mathcal{V}_4(\ell_4)}{\partial\mathcal{V}_4(\ell_4^i)} = \mathbb{V}(\ell_4) + \int_{\ell_4^i}^{\ell_4} \mathbb{V}(\ell_4 - s) \sum_{j=3,1} \frac{\partial\mathcal{F}_4}{\partial\mathcal{V}_j} \cdot \frac{\partial\mathcal{V}_j(s)}{\partial\mathcal{V}_4(\ell_4^i)} ds,$$

where \mathbb{V} is the fundamental solution of the homogeneous equation $\frac{dX}{d\ell_4} = \frac{\partial\mathcal{F}_4}{\partial\mathcal{V}_4}X$.

First we know that $\frac{\partial\mathcal{V}_4(\ell_4)}{\partial\mathcal{V}_4(\ell_4^i)}, \mathbb{V} = O(1)$ from the 44 block of the matrix M, N_1, N_5 .

The blocks for $\frac{\partial\mathcal{V}_j(s)}{\partial\mathcal{V}_4(\ell_4^i)}$, $j = 3, 1$ in M, N are bounded by μ in the right case

and by $O(1/\chi)$ in the left case. Moreover, the integral $\int_{\ell_4^i}^{\ell_4^f} \frac{\partial\mathcal{F}_4}{\partial\mathcal{V}_j} d\ell_4$, $j = 3, 1$ is

bounded by $O(\mu)$ in the right case and $O(1/\chi)$ in the left case. As a result, we get $\frac{\partial\mathcal{V}_4(\ell_4)}{\partial\mathcal{V}_4(\ell_4^i)} = \mathbb{V}(\ell_4) + o(1)$ as $1/\chi \ll \mu \rightarrow 0$.

We need to find the asymptotics of \mathbb{V} . Consider map $(N_1)_{44}$ first. \mathbb{V} satisfies

$$\mathbb{V}' = \frac{\partial\mathcal{F}}{\partial\mathcal{V}_4}\mathbb{V}, \quad \mathbb{V}' = \frac{\xi L^2}{\chi(1-\xi)^3}A\mathbb{V} + O\left(\frac{\mu}{\ell_4^2+1} + \frac{\mu}{\chi}\right).$$

where $A = \begin{bmatrix} -\frac{L^2}{(G^2+L^2)} & L \\ -\frac{L^3}{(G^2+L^2)^2} & \frac{L^2}{(G^2+L^2)} \end{bmatrix}$ following from part (c) of Lemma 7.2. Now

Gronwall Lemma gives $\mathbb{V} \approx \tilde{\mathbb{V}}$ where $\tilde{\mathbb{V}}$ is the fundamental solution of $\tilde{\mathbb{V}}' = \frac{\xi L^2}{\chi(1-\xi)^3}A\tilde{\mathbb{V}}$. Using ξ as the independent variable we get $\frac{d\tilde{\mathbb{V}}}{d\xi} = -\frac{\xi}{(1-\xi)^3}A\tilde{\mathbb{V}}$.

Note that $\xi(\ell_4^i) = o(1)$, $\xi(\ell_4^f) = \frac{1}{2} + o(1)$. Making a further time change $d\tau =$

$\frac{\xi d\xi}{(1-\xi)^3}$ we obtain the constant coefficient linear equation $\frac{d\tilde{\mathbb{V}}}{d\tau} = -A\tilde{\mathbb{V}}$. Observe that $\text{Tr}(A) = \det(A) = 0$ and so $A^2 = 0$. Therefore

$$(7.16) \quad \tilde{\mathbb{V}}(\sigma, \tau) = \text{Id} - (\tau - \sigma)A.$$

Since $\tau = \frac{\xi^2}{2(1-\xi)^2}$ we have $\tau(0) = 0$, $\tau\left(\frac{1}{2}\right) = \frac{1}{2}$. Plugging this into (7.16) we get the claimed asymptotics for $(N_1)_{44}$. The analysis of map $(N_5)_{44}$ is similar. To analyze $(M)_{44}$ we split

$$\frac{\partial\mathcal{V}_4(\ell_4^f)}{\partial\mathcal{V}_4(\ell_4^i)} = \frac{\partial\mathcal{V}_4(\ell_4^f)}{\partial\mathcal{V}_4(\ell_4^m)} \frac{\partial\mathcal{V}_4(\ell_4^m)}{\partial\mathcal{V}_4(\ell_4^i)}$$

where $\ell_4^m = \frac{\ell_4^i + \ell_4^f}{2}$. Using the argument presented above we obtain

$$\frac{\partial\mathcal{V}_4(\ell_4^m)}{\partial\mathcal{V}_4(\ell_4^i)} = \begin{bmatrix} \frac{3}{2} & -\frac{L}{2} \\ \frac{1}{2L} & \frac{1}{2} \end{bmatrix}, \quad \frac{\partial\mathcal{V}_4(\ell_4^f)}{\partial\mathcal{V}_4(\ell_4^m)} = \begin{bmatrix} \frac{1}{2} & -\frac{L}{2} \\ \frac{1}{2L} & \frac{3}{2} \end{bmatrix}.$$

Multiplying the above matrices we obtain the required asymptotics $(M)_{44}$.

□

8. ESTIMATES OF THE BOUNDARY CONTRIBUTION

In general it takes different times for different orbits to move between two consecutive sections. For this reason, the solution of variational equation is not the derivative of the Poincaré map. We need to take into account of the boundary contributions. In this section, we first derive a formula giving us the correct derivative of the Poincaré map as well as the explicit expression for obtaining the boundary contribution. Next, we work on all the boundary contributions for the map (I), (III), (V).

8.1. Derivation of the formula for computing the boundary contribution.

Suppose that we want to compute the derivative of the Poincaré map between the sections S^i and S^f . We use \mathcal{V}^i to denote the values of variables \mathcal{V} restricted on the *initial* section S^i , while \mathcal{V}^f means values of \mathcal{V} on the *final* section S^f . ℓ_4^i means the initial time and ℓ_4^f means the final time. We want to compute the derivative \mathcal{D} of the Poincaré map along the orbit starting from $(\mathcal{V}_*, \ell_4^i)$ and ending at $(\mathcal{V}_*, \ell_4^f)$. We have $\mathcal{D} = dF_3 dF_2 dF_1$ where F_1 is the Poincaré map between S^i and $\{\ell_4 = \ell_4^i\}$, F_2 is the flow map between the times ℓ_4^i and ℓ_4^f , and F_3 is the Poincaré map between $\{\ell_4 = \ell_4^f\}$ and S^f . We have $F_1 = \Phi(\mathcal{V}^i, \ell_4(\mathcal{V}^i), \ell_4^i)$ where $\Phi(\mathcal{V}, a, b)$ denotes the flow map starting from \mathcal{V} at time a and ending at time b . Since

$$\frac{\partial \Phi}{\partial \mathcal{V}}(\mathcal{V}_*, \ell_4^i, \ell_4^i) = \text{Id}, \quad \frac{\partial \Phi}{\partial a} = -\mathcal{F}$$

we have $dF_1 = \text{Id} - \mathcal{F}(\ell_4^i) \otimes \frac{D\ell_4^i}{D\mathcal{V}^i}$. Inverting the time we get $dF_3 = \left(\text{Id} - \mathcal{F}(\ell_4^f) \otimes \frac{D\ell_4^f}{D\mathcal{V}^f} \right)^{-1}$.

Finally $dF_2 = \frac{D\mathcal{V}(\ell_4^f)}{D\mathcal{V}(\ell_4^i)}$ is just the fundamental solution of the variational equation between the times ℓ_4^i and ℓ_4^f . Thus we get

$$(8.1) \quad \mathcal{D} = \left(\text{Id} - \mathcal{F}(\ell_4^f) \otimes \frac{D\ell_4^f}{D\mathcal{V}^f} \right)^{-1} \frac{D\mathcal{V}(\ell_4^f)}{D\mathcal{V}(\ell_4^i)} \left(\text{Id} - \mathcal{F}(\ell_4^i) \otimes \frac{D\ell_4^i}{D\mathcal{V}^i} \right).$$

We call the two terms dF_1 , dF_3 the boundary contributions.

8.2. Boundary contribution for (I).

Computation of matrix (I) in Proposition 5.1. By (8.1) (I) is a product of three matrices (8.1) and we already know the matrix N_1 , i.e. the solution of the variational equation. It remains to work out the two matrices for boundary contributions. The expression for $x_{4,\parallel}^R$ is the following (see Appendix A)

$$(8.2) \quad x_{4,\parallel}^R = \cos g_4(L_4^2 \sinh u_4 - e_4) - \sin g_4(L_4 G_4 \cosh u_4).$$

For fixed $x_{4,\parallel}^R = -\chi/2$ or -2 , we can solve ℓ_4 as a function of L_4, G_4, g_4 . The bounds for L_4, G_4 have been obtained in Lemma 6.2(a). So we get the following

using implicit function theorem and Lemma A.2.

$$(8.3) \quad \left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4} \right) = -\frac{1}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}} \left(\frac{\partial x_{4,\parallel}}{\partial L_4}, \frac{\partial x_{4,\parallel}}{\partial G_4}, \frac{\partial x_{4,\parallel}}{\partial g_4} \right).$$

$$\left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4} \right)^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim (\chi, \mu \mathcal{G}, \mu \mathcal{G}), \quad \left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4} \right)^R \Big|_{x_{4,\parallel}^R = -2} \lesssim (1, 1, 1).$$

Using Corollary 6.2, and Lemma 7.1, we obtain

for the section, $x_{4,\parallel}^R = -2$,

$$(8.4) \quad \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -2} \lesssim \left(1, \mu, \mu, \mu; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; 1, 1 \right),$$

$$\mathcal{F}^R \Big|_{x_{4,\parallel}^R = -2} \lesssim \left(\mu, 1, \mu, \mu; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \mu, \mu \right)^T.$$

and for the section $x_{4,\parallel}^R = -\chi/2$,

$$(8.5) \quad \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim \left(\chi, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi^2}; \frac{1}{\mu \chi}, \frac{\mathcal{G}}{\chi^2}, \mu \chi, \mu \mathcal{G}; \mu \mathcal{G}, \mu \mathcal{G} \right),$$

$$\mathcal{F}^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim \left(\frac{1}{\chi^3}, \mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2} \right)^T,$$

where the two entries in bold fonts are estimates in the sense of \sim rather than O .

The $\mathbf{1}$ entry in $\mathcal{F}^R \Big|_{x_{4,\parallel}^R = -\chi/2}$ is already established in Corollary 6.2. To get the

χ entry in $\left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -\chi/2}$, we use the $(1, 1), (1, 2)$ entries in \mathcal{D} from Lemma

A.7. The result is $-\frac{\frac{\partial L_4}{\partial x_{4,\parallel}}}{\frac{\partial \ell_4}{\partial x_{4,\parallel}}} = -2\ell/L = \chi/L^3$, where the last equality is obtained by setting $Q_{\parallel} = -\chi/2$ in (A.5). In this case $u > 0, \ell_4 < 0$. Denote

$$(8.6) \quad l := \left(\mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; \frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi} \right),$$

$$u := \left(\frac{1}{\chi^3}, \mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2} \right)^T.$$

Then (8.5) gives

$$(8.7) \quad \frac{1}{\chi} \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim l, \quad \mathcal{F}^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim u.$$

Define

$$(8.8) \quad u_1^i = \mathcal{F}^R \Big|_{x_{4,\parallel}^R = -2}, \quad l_1^i = \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -2},$$

$$u_1^f = \mathcal{F}^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim u, \quad l_1^f = \frac{1}{\chi} \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim l,$$

where the inequities follow from (8.7) and (8.5). Then $(I) = (\text{Id} + \chi u_1^f \otimes l_1^f)^{-1} N_1 (\text{Id} + u_1^i \otimes l_1^i)$ as claimed in Proposition 5.1.

The matrix $\mathcal{F}^R \otimes \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -\chi/2}$ has rank 1 and the only nonzero eigenvalue is $O(1/\chi)$, and $\mathcal{F}^R \otimes \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -2}$ has rank 1 and the only nonzero eigenvalue is $O(\mu)$. So the inversion appearing in (8.1) is valid. To invert $\text{Id} - \mathcal{F}^R \otimes \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R$, we use

$$\begin{aligned} \left(\text{Id} - \mathcal{F}^R \otimes \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \right)^{-1} &= \text{Id} + \sum_{n \geq 1} \left(\mathcal{F}^R \otimes \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \right)^n \\ (8.9) \qquad \qquad \qquad &= \text{Id} + \mathcal{F}^R \otimes \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \sum_{n \geq 0} \left(\left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \cdot \mathcal{F}^R \right)^n. \end{aligned}$$

In both cases of $x_{4,\parallel}^R = -\chi/2$ and -2 , the series $\sum_{n \geq 0} \left(\left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \cdot \mathcal{F}^R \right)^n$ converges to $1 + o(1)$.

Finally, we show the β rotation of the section $\{x_{4,\parallel}^R = -2, v_{4,\parallel}^R > 0\}$ to the section $\{(\text{Rot}(-\beta) \cdot x_4)_{\parallel}^R = -2, v_{4,\parallel}^R > 0\}$ after applying \mathcal{R} in Definition 2.3 is negligible. Instead of (8.2), we need to use the expression $\cos \beta \cdot x_{4,\parallel}^R - \sin \beta \cdot x_{4,\perp}^R = -2$ and convert x_4^R into Delaunay variables. Since we have $\ell_4^R = O(1)$ here, and $\beta = O(\mu \mathcal{G}/\chi)$ according to part (c) of Lemma 6.2, we get a correction of order $O(\mu \mathcal{G}/\chi) \cdot \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R$ to $\left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R$ in (8.4), which is negligible. \square

8.3. Boundary contribution for (III).

Computation of matrix (III) in Proposition 5.1. For the matrix (III), the solution for the variational equation is given by M . We only need to work out the two boundary terms on the sections $\left\{ x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0 \right\}$, $\left\{ x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0 \right\}$.

In (8.1), the variables $\mathcal{V}^i, \mathcal{V}^f$ should carry superscript L for matrix (III) since we did not compose a coordinates change between the left and right variables in (8.1). However, the section $\left\{ x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0 \right\}$ is defined using variables with superscript R , we need first to express it using left variables. We use the matrix $R \cdot L^{-1}$ to get $\mathcal{X}^R = R \cdot L^{-1} \mathcal{X}^L$. This implies

$$\begin{aligned} x_{4,\parallel}^R &= x_{1,\parallel}^L + \frac{1+\mu}{1+2\mu} x_{4,\parallel}^L = \frac{\chi}{2}, \\ x_{4,\parallel}^R &= x_{1,\parallel}^L + \frac{1+\mu}{1+2\mu} (\cos g_4 (L_4^2 \sinh u_4 - e_4) - \sin g_4 (L_4 G_4 \cosh u_4)) = -\frac{\chi}{2}. \end{aligned}$$

So we get the following using implicit function theorem and Appendix Lemma A.2.

$$(8.11) \quad \left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4}, \frac{\partial \ell_4}{\partial x_{1,\parallel}} \right)^L = - \frac{1}{\frac{\partial x_{4,\parallel}^L}{\partial \ell_4}} \left(\frac{\partial x_{4,\parallel}}{\partial L_4}, \frac{\partial x_{4,\parallel}}{\partial G_4}, \frac{\partial x_{4,\parallel}}{\partial g_4}, \frac{1+2\mu}{1+\mu} \right)^L,$$

$$\left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4}, \frac{\partial \ell_4}{\partial x_{1,\parallel}} \right)^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim (\chi, \mu \mathcal{G}, \mu \mathcal{G}, 1).$$

Using Corollary 6.2 and Lemma 7.1, we obtain

$$(8.12) \quad \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim \left(\chi, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi^2}; 1, \frac{\mathcal{G}}{\chi^2}, \mu \chi, \mu \mathcal{G}; \mu \mathcal{G}, \mu \mathcal{G} \right),$$

$$\mathcal{F}^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim \left(\frac{1}{\chi^3}, \mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \mu, \frac{\mu \mathcal{G}}{\chi}, \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2} \right)^T,$$

where $\mathbf{1}$ and χ are estimates in the sense of \sim , having the same values as that in (8.5). Denote

$$(8.13) \quad l' := \left(\mathbf{1}, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\chi}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}; \frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi} \right),$$

which is different from l in its fifth entry. Then (8.12) becomes

$$(8.14) \quad \frac{1}{\chi} \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim l', \quad \mathcal{F}^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim u.$$

For the section $\left\{ x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0 \right\}$, the estimate is exactly the same as the case $\left\{ x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^L < 0 \right\}$ in (I), i.e. u_1^f and l_1^f , and we get the same result as (8.8)

$$(8.15) \quad u_3^f := \mathcal{F}^L \Big|_{x_{4,\parallel}^L = \chi/2} \lesssim u, \quad l_3^f := \frac{1}{\chi} \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^L \Big|_{x_{4,\parallel}^L = \chi/2} \lesssim l.$$

We obtain the matrix $(III) = (\text{Id} + \chi u_3^f \otimes l_3^f)^{-1} M (\text{Id} + \chi u_3^i \otimes l_3^i)$ in Proposition 5.1 by defining

$$(8.16) \quad l_3^i := \frac{1}{\chi} \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim l', \quad u_3^i := \mathcal{F}^L \Big|_{x_{4,\parallel}^R = -\chi/2} \lesssim u,$$

where the inequalities follow from (8.14). \square

8.4. Boundary contribution for (V).

Computation of matrix (V) in Proposition 5.1. For the matrix (V), the solution of the variational equation is given by N_5 . We only need to get two boundary contributions. Notice the section $\left\{ x_{4,\parallel}^L = \frac{\chi}{2}, v_{4,\parallel}^L > 0 \right\}$ is defined using left variables. However, we need to express the boundary contributions in (8.1). The estimate is exactly the same as that for the section $\left\{ x_{4,\parallel}^R = -\frac{\chi}{2}, v_{4,\parallel}^R < 0 \right\}$ of (III), i.e. u_3^i and l_3^i , though this time we need to use $\mathcal{X}^L = L \cdot R^{-1} \mathcal{X}^R$. We get the same result as (8.16)

$$(8.17) \quad u_5^i := \mathcal{F}^R \Big|_{x_{4,\parallel}^L = \chi/2} \lesssim u, \quad l_5^i := \frac{1}{\chi} \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^L = \chi/2} \lesssim l'.$$

For the section $\{x_{4,\parallel}^R = -2\}$, the estimate is exactly the same as the estimate in the $\{x_{4,\parallel}^R = 2\}$ case of (I), i.e. u_1^i and l_1^i in (8.8). Defining

$$(8.18) \quad u_5^f := \mathcal{F}^R \Big|_{x_{4,\parallel}^R = -2}, \quad l_5^f := \left(\frac{\partial \ell_4}{\partial \mathcal{V}} \right)^R \Big|_{x_{4,\parallel}^R = -2}$$

we get $(V) = (\text{Id} + \chi u_5^f \otimes l_5^f)^{-1} N_5 (\text{Id} + u_1^i \otimes l_1^i)$ as claimed in Proposition 5.1. \square

9. ESTIMATES OF THE MATRICES FOR SWITCHING FOCI

In this section, we study the matrices (II) and (IV) in Proposition 5.1.

9.1. A simplifying computation. We start with a formal calculation, which liberates us from calculating the \mathcal{V}_3 part. Both $R \cdot L^{-1}$ and $L \cdot R^{-1}$ can be represented as $\begin{bmatrix} \text{Id}_{4 \times 4} & 0 \\ 0 & T_\mu \end{bmatrix}$ for a 8×8 matrix T_μ . We need to multiply to the left $\frac{\partial \mathcal{V}^L}{\partial \mathcal{X}^L}$ and to the right $\frac{\partial \mathcal{X}^R}{\partial \mathcal{V}^R}$ to get (II) = $\frac{\partial \mathcal{V}^L}{\partial \mathcal{V}^R}$ as follows

$$(9.1) \quad \begin{aligned} & \frac{\partial \mathcal{V}^L}{\partial \mathcal{X}^L} L \cdot R^{-1} \frac{\partial \mathcal{X}^R}{\partial \mathcal{V}^R} \\ &= \begin{bmatrix} \frac{\partial \mathcal{V}_3}{\partial \mathcal{X}_3} & 0 \\ 0 & \frac{\partial(\mathcal{V}_1, \mathcal{V}_4)}{\partial(\mathcal{X}_1, \mathcal{X}_4)} \end{bmatrix}_{10 \times 12}^L \begin{bmatrix} \text{Id}_{4 \times 4} & 0 \\ 0 & T_\mu \end{bmatrix}_{12 \times 12} \begin{bmatrix} \frac{\partial \mathcal{X}_3}{\partial \mathcal{V}_3} & 0 \\ \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial \mathcal{V}_3} & \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial(\mathcal{V}_1, \mathcal{V}_4)} \end{bmatrix}_{12 \times 10}^R \\ &= \begin{bmatrix} \text{Id} & 0 \\ \frac{\partial(\mathcal{V}_1, \mathcal{V}_4)^L}{\partial(\mathcal{X}_1, \mathcal{X}_4)^L} T_\mu \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}{\partial \mathcal{V}_3^R} & \frac{\partial(\mathcal{V}_1, \mathcal{V}_4)^L}{\partial(\mathcal{X}_1, \mathcal{X}_4)^L} T_\mu \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}{\partial(\mathcal{V}_1, \mathcal{V}_4)^R} \end{bmatrix}_{10 \times 10} \\ &= \begin{bmatrix} \text{Id} & 0 \\ 0 & 0 \end{bmatrix}_{10 \times 10} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial(\mathcal{V}_1, \mathcal{V}_4)}{\partial(\mathcal{X}_1, \mathcal{X}_4)} \end{bmatrix}_{10 \times 12}^L \begin{bmatrix} 0 & 0 \\ 0 & T_\mu \end{bmatrix}_{12 \times 12} \begin{bmatrix} 0 & 0 \\ \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial \mathcal{V}_3} & \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial(\mathcal{V}_1, \mathcal{V}_4)} \end{bmatrix}_{12 \times 10}^R \\ &= \begin{bmatrix} \text{Id} & 0 \\ 0 & \frac{\partial(\mathcal{V}_1, \mathcal{V}_4)}{\partial(\mathcal{X}_1, \mathcal{X}_4)} \end{bmatrix}_{10 \times 12}^L \begin{bmatrix} \text{Id} & 0 \\ 0 & T_\mu \end{bmatrix}_{12 \times 12} \begin{bmatrix} \text{Id} & 0 \\ \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial \mathcal{V}_3} & \frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial(\mathcal{V}_1, \mathcal{V}_4)} \end{bmatrix}_{12 \times 10}^R. \end{aligned}$$

We have the same calculation for (IV) = $\frac{\partial \mathcal{V}^R}{\partial \mathcal{V}^L} = \frac{\partial \mathcal{V}^R}{\partial \mathcal{X}^R} R \cdot L^{-1} \frac{\partial \mathcal{X}^L}{\partial \mathcal{V}^L}$. In the following, we only need to figure out the matrices $\frac{\partial(\mathcal{X}_1, \mathcal{X}_4)}{\partial(\mathcal{V}_3, \mathcal{V}_1, \mathcal{V}_4)}$ and $\frac{\partial(\mathcal{V}_1, \mathcal{V}_4)}{\partial(\mathcal{X}_1, \mathcal{X}_4)}$.

9.2. From Delaunay to Cartesian coordinates. In this section we compute

$$\frac{\partial(\mathcal{X}_1; \mathcal{X}_4)}{\partial(\mathcal{V}_3, \mathcal{V}_1, \mathcal{V}_4)} = \frac{\partial(x_1, v_1, x_4, v_4)}{\partial(L_3, \ell_3, G_3, g_3, x_1, v_1, G_4, g_4)}.$$

This computation is done restricted on the section $\{x_{4,\parallel}^L = -\chi/2\}$ for matrix (II) and on the section $\{x_{4,\parallel}^L = \chi/2\}$ for matrix (IV). The key observation to obtain the tensor structure of the following sublemma is explained in Remark A.1 (2).

Sublemma 9.1. *Assume (6.6), (6.25), then*

(a) on the section $\{x_{4,\parallel}^R = -\chi/2\}$ the matrix $\frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}{\partial \mathcal{V}^R}$ in (9.1) is a 8×10 matrix of the form

$$(9.2) \quad \frac{\partial(x_1, v_1, x_4, v_4)^R}{\partial(L_3, \ell_3, G_3, g_3, x_1, v_1, G_4, g_4)^R} = \chi u_i \otimes l_i + \left[\begin{array}{cc|cc} 0_{4 \times 4} & \text{Id}_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} \\ 0_{1 \times 4} & 0_{1 \times 4} & 0 & 0 \\ 0_{1 \times 4} & 0_{1 \times 4} & O(1) & O(1) \\ \check{l}_i & & O\left(\frac{\mu \mathcal{G}}{\chi}\right) & O\left(\frac{\mu \mathcal{G}}{\chi}\right) \\ 0_{1 \times 4} & 0_{1 \times 4} & 0 & 0 \end{array} \right]_{8 \times 10}$$

where we have the estimates

$$u_i = \left(0_{1 \times 5}, \frac{L_4}{2m_4^2 k_4^2}, 0, \frac{1}{\chi} \right)^T, \quad \check{l}_i \lesssim \left(1, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi} \right),$$

$$l_i = \left(\frac{-G_4 k_4 m_4}{L_4(G_4^2 + L_4^2)}, O\left(\frac{1}{\chi^3}\right)_{1 \times 3}; O\left(\frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi}\right); \frac{k_4 m_4}{G_4^2 + L_4^2}, \frac{-k_4 m_4}{L_4} \right)^R_{1 \times 10}$$

and l_i converges to $\hat{\mathbf{l}}$ defined in Lemma 3.2 as $1/\chi \ll \mu \rightarrow 0$.

(b) On the section $\{x_{4,\parallel}^L = \chi/2\}$ the matrix $\frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^L}{\partial \mathcal{V}^L}$ for (IV) has the same form with the same u_i and l_i replaced by

$$l_{i'} = \left(0_{1 \times 8}, \frac{k_4 m_4}{L_4^2}, \frac{-k_4 m_4}{L_4} \right) + O\left(\frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi^4}, \frac{\mu \mathcal{G}}{\chi^4}, \frac{\mu \mathcal{G}}{\chi^4}; \frac{\mathcal{G}}{\chi^3}, \frac{\mu \mathcal{G}^2}{\chi^4}, \frac{\mu^2 \mathcal{G}}{\chi}, \frac{\mu^2 \mathcal{G}^2}{\chi^2}; \frac{\mathcal{G}}{\chi^3}, 0 \right).$$

Proof. We trivially have $\left(\frac{\partial \mathcal{X}_1}{\partial(\mathcal{V}_3, \mathcal{V}_4)} \right)^{R,L} = 0$, and $\left(\frac{\partial \mathcal{X}_1}{\partial \mathcal{V}_1} \right)^{R,L} = \text{Id}_4$ since the variables $\mathcal{X}_1 = (x_1, v_1)$ are not transformed to Delaunay variables and they are independent of $\mathcal{V}_{3,4}$. It remains to obtain $\frac{\partial \mathcal{X}_4}{\partial \mathcal{V}}$.

Step 1, formal derivations.

In the following calculation, we use (8.11). The formal calculation works for both cases, left and right, so we omit the superscripts. Variables L_4, ℓ_4 are eliminated from the list of variables, so they need to be paid special attention.

$$\begin{aligned}
(9.3) \quad \frac{\partial \mathcal{X}_4}{\partial \mathcal{V}} &= \frac{\partial \mathcal{X}_4}{\partial(L_4, \ell_4)} \frac{\partial(L_4, \ell_4)}{\partial \mathcal{V}} + \left(0_{4 \times 8} \left| \frac{\partial \mathcal{X}_4}{\partial(G_4, g_4)} \right. \right) \\
&= \left(\frac{\partial \mathcal{X}_4}{\partial L_4}, \frac{\partial \mathcal{X}_4}{\partial \ell_4} \right) \left(\frac{\partial L_4}{\partial \mathcal{V}}; \frac{\partial \ell_4}{\partial \mathcal{V}} \right) + \left(0_{4 \times 8} \left| \frac{\partial \mathcal{X}_4}{\partial G_4}, \frac{\partial \mathcal{X}_4}{\partial g_4} \right. \right) \\
&= \left(\frac{\partial \mathcal{X}_4}{\partial L_4} + \frac{\partial \ell_4}{\partial L_4} \frac{\partial \mathcal{X}_4}{\partial \ell_4} \right) \otimes \frac{\partial L_4}{\partial \mathcal{V}} + \frac{\partial \mathcal{X}_4}{\partial \ell_4} \otimes \left(0_{1 \times 8}; \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4} \right) + \left(0_{4 \times 8} \left| \frac{\partial \mathcal{X}_4}{\partial G_4}, \frac{\partial \mathcal{X}_4}{\partial g_4} \right. \right) \\
&= \left(\frac{\partial \mathcal{X}_4}{\partial L_4} - \frac{\frac{\partial x_{4,\parallel}}{\partial L_4} \frac{\partial \mathcal{X}_4}{\partial \ell_4}}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}} \right) \otimes \frac{\partial L_4}{\partial \mathcal{V}} - \frac{1}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}} \frac{\partial \mathcal{X}_4}{\partial \ell_4} \otimes \left(0_{1 \times 8}; \frac{\partial x_{4,\parallel}}{\partial G_4}, \frac{\partial x_{4,\parallel}}{\partial g_4} \right) + \\
&\quad \left(0_{4 \times 8} \left| \frac{\partial \mathcal{X}_4}{\partial G_4}, \frac{\partial \mathcal{X}_4}{\partial g_4} \right. \right) \\
&= \left(\frac{\partial \mathcal{X}_4}{\partial L_4} - \frac{\frac{\partial x_{4,\parallel}}{\partial L_4} \frac{\partial \mathcal{X}_4}{\partial \ell_4}}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}} \right) \otimes \frac{\partial L_4}{\partial \mathcal{V}} + \left(0_{4 \times 8} \left| \frac{\partial \mathcal{X}_4}{\partial G_4} - \frac{\frac{\partial x_{4,\parallel}}{\partial G_4} \frac{\partial \mathcal{X}_4}{\partial \ell_4}}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}}, \frac{\partial \mathcal{X}_4}{\partial g_4} - \frac{\frac{\partial x_{4,\parallel}}{\partial g_4} \frac{\partial \mathcal{X}_4}{\partial \ell_4}}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}} \right. \right) \\
&\sim \left(0, \frac{G_4 L_4^2 \ell_4}{m_4 k_4 (G_4^2 + L_4^2)}, \frac{k_4 m_4}{L_4^2}, -\frac{G_4 k_4 m_4}{L_4 (G_4^2 + L_4^2)} \right) \otimes \frac{\partial L_4}{\partial \mathcal{V}} + \\
&\quad \left(0_{4 \times 8} \left| \begin{array}{cc} 0 & 0 \\ \frac{L_4^3 \ell_4 / m_4 k_4}{(G_4^2 + L_4^2)} & \frac{-L_4^2 \ell_4}{m_4 k_4} \\ \frac{\mu \mathcal{G}}{\mu \mathcal{G}} & \frac{\mu \mathcal{G}}{\mu \mathcal{G}} \\ \frac{\chi}{k_4 m_4} & \frac{\chi}{k_4 m_4} \\ \frac{G_4^2 + L_4^2}{L_4} & -\frac{L_4}{L_4} \end{array} \right. \right)_{4 \times 10}
\end{aligned}$$

where in the last step, we use Lemma A.3, and choose the upper sign when we need to make a choice in \pm and \mp . Actually, the terms $\frac{\frac{\partial x_{4,\parallel}}{\partial(L_4, G_4, g_4)} \frac{\partial \mathcal{X}_4}{\partial \ell_4}}{\frac{\partial x_{4,\parallel}}{\partial \ell_4}}$ are small compared to the corresponding $\frac{\partial \mathcal{X}_4}{\partial(L_4, G_4, g_4)}$ due to the smallness of $\frac{\partial(x_{4,\perp}, v_4)}{\partial \ell_4}$ in (A.7).

It is easy to see from the above calculation that the first row is zero since the first entry of \mathcal{X}_4 is $x_{4,\parallel}$. This also follows from the fact that we are restricted on the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$ so that $x_{4,\parallel}$ is a constant. We already have the tensor structure in the first summand of the last line of (9.3). We also want to get a rank 1 structure in the second summand. To this end we note that in equation (A.7) the two vectors $\frac{\partial x_{4,\perp}}{\partial(L_4, G_4, g_4)}$ and $\frac{\partial v_{4,\perp}}{\partial(L_4, G_4, g_4)}$ (the second and fourth rows in (A.7)) are parallel with ratio of modulus $\frac{L_4^3 \ell_4}{m_4^2 k_4^2}$ if we discard the $O(1)$ terms in the former (see Remark A.1).

Step 2, the case $\frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}{\partial \mathcal{V}^R}$ on the section $\{x_{4,\parallel}^R = -\chi/2\}$.

Orbit parameters in this step should carry a superscript R which we omit for simplicity. We define the last row of the above calculation (9.3) as the vector l_i . That is,

$$(9.4) \quad -l_i := \left(\frac{G_4 k_4 m_4}{L_4(G_4^2 + L_4^2)} \right) \cdot \frac{\partial L_4}{\partial \mathcal{V}} + \left(0_{1 \times 8}; \frac{-k_4 m_4}{G_4^2 + L_4^2}, \frac{k_4 m_4}{L_4} \right)$$

We get the estimate of l_i stated in the lemma using Lemma 7.1 for the section $\{x_{4,\parallel}^R = -\chi/2\}$. Moreover, since the first entry in $\frac{\partial L_4}{\partial \mathcal{V}}$ is $\frac{\partial L_4}{\partial L_3} = 1 + O(\mu)$ and the last two entries are $O(\mu \mathcal{G}/\chi^2)$, we see that $l_i \rightarrow \hat{\mathbf{l}}$ defined in Lemma 3.2 when we take limit $1/\chi \ll \mu \rightarrow 0$.

Then the second row of the last line of (9.3) is $\frac{L_4^3 \ell_4}{m_4^2 k_4^2} l_i + (0_{1 \times 8}; O(1), O(1))$. For the third row, we define a vector $\check{l}_i = \left(1, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{1}{\mu \chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu \mathcal{G}}{\chi} \right)$ as the first 8 entries of l_i , so the third row is $(\check{l}_i; \mu \mathcal{G}/\chi, \mu \mathcal{G}/\chi)$. We enlarge the vector $\left(0, \frac{G_4 L_4^2 \ell_4}{m_4 k_4 (G_4^2 + L_4^2)}, 0, -\frac{G_4 k_4 m_4}{L_4 (G_4^2 + L_4^2)} \right) / \left(-\chi \frac{G_4 k_4 m_4}{L_4 (G_4^2 + L_4^2)} \right)$ to the following

$$u_i = \left(0, 0, 0, 0; 0, \frac{L_4}{2m_4^2 k_4^2}, 0, \frac{1}{\chi} \right)^T,$$

where we used (A.5) to get that $L_4^2 \ell_4 \simeq -\frac{\chi}{2}$ when restricted to the section $\{x_{4,\parallel}^R = -\frac{\chi}{2}\}$.

This completes the proof in the case $\frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}{\partial \mathcal{V}^R}$.

The case of $\frac{\partial(\mathcal{X}_1, \mathcal{X}_4)^L}{\partial \mathcal{V}^L}$ on the section $\{x_{4,\parallel}^L = \chi/2\}$. This case follows from the same formal calculation (9.3). However, since the variables are to the left of the section $x_{4,\parallel}^L = \chi/2$, we have $G_4^L = O\left(\frac{\mu \mathcal{G}}{\chi}\right)$ in (9.4) according to Lemma 6.9(b).

Thus we get an improved l_i in place of l_i by applying Lemma 7.1 to (9.4). We also have $L_4^2(-\ell_4) \simeq \chi/2$ using (A.6) for $u > 0, \ell_4 < 0$ to the left of the section $x_{4,\parallel}^L = \chi/2$. So u_i in the left case has the same expression as in the right case. \square

9.3. From Cartesian to Delaunay coordinates. In this section we compute $\frac{\partial(\mathcal{V}_1; \mathcal{V}_4)}{\partial(\mathcal{X}_1, \mathcal{X}_4)} = \frac{\partial(x_1, v_1, G_4, g_4)}{\partial(x_1, v_1, x_4, v_4)}$. The key observation to get the tensor structure is explained in Remark A.1 (3).

Sublemma 9.2. Assume (6.6), (6.25), then

(a) on the section $\{x_{4,\parallel}^R = -\chi/2\}$ the matrix $\frac{\partial(\mathcal{V}_1; \mathcal{V}_4)^L}{\partial(\mathcal{X}_1, \mathcal{X}_4)^L}$ in (9.1) is a 6×8 matrix of the following form

$$(9.5) \quad \frac{\partial(x_1, v_1, G_4, g_4)^L}{\partial(x_1, v_1, x_4, v_4)^L} = \chi u_{iii} \otimes l_{iii} + \left[\begin{array}{c|cccc} \text{Id}_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ \hline 0_{1 \times 4} & 0 & 0 & 0 & 0 \\ 0_{1 \times 4} & O\left(\frac{1}{\chi^2}\right) & \left(\frac{1}{\chi^2}\right) & O(1) & O(1) \end{array} \right]_{6 \times 8},$$

where we have estimates

$$u_{iii} = \left(0, 0, 0, 0; 1, \frac{1}{L_4} + O\left(\frac{\mu^2 \mathcal{G}^2}{\chi^2}\right)\right)^T, \quad l_{iii} = \left(0_{1 \times 4}; O\left(\frac{\mu \mathcal{G}}{\chi^2}\right), \frac{-m_4 k_4}{\chi L_4}, O\left(\frac{\mu \mathcal{G}}{\chi}\right), -\frac{1}{2}\right).$$

(b) On the section $\{x_{4,\parallel}^L = \chi/2\}$ the matrix $\frac{\partial(\mathcal{V}_1; \mathcal{V}_4)^R}{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}$ for (IV) has the same form with

$$u_{iii'} = \left(0, 0, 0, 0; 1, -\frac{L_4}{G_4^2 + L_4^2}\right)^T$$

and l_{iii} replaced by $l_{iii'} = -l_{iii}$.

Proof. The only nontrivial part of this matrix is $\frac{\partial(G_4, g_4)}{\partial(x_4, v_4)}$. We consider first part (a), to the left of the section $\{x_{4,\parallel}^R = -\chi/2\}$. It follows from Lemma A.3(b) that

$$\frac{\partial g_4}{\partial(x_4, v_4)} = \frac{L_4}{G_4^2 + L_4^2} \frac{\partial G_4}{\partial(x_4, v_4)} + O\left(\frac{1}{\chi^2}, \frac{1}{\chi^2}, 1, 1\right).$$

This implies that the two rows in $\frac{\partial(G_4, g_4)^L}{\partial(x_4, v_4)^L}$ are almost parallel up to the O term.

Then we define

$$u_{iii} = \left(0_{1 \times 4}; 1, \frac{L_4}{G_4^2 + L_4^2}\right)^T, \\ l_{iii} = \frac{1}{\chi} \left(0_{1 \times 4}; \frac{\partial G_4}{\partial(x_4, v_4)}\right) = \left(0_{1 \times 4}; O\left(\frac{\mu \mathcal{G}}{\chi^2}\right), \frac{-m_4 k_4}{\chi L_4}, O\left(\frac{\mu \mathcal{G}}{\chi}\right), -\frac{1}{2}\right),$$

where the entry $\frac{-m_4 k_4}{\chi L_4}$ is obtained using the following formulas

$$\frac{\partial G}{\partial Q_\perp} = P_\parallel \text{ (by Lemma A.3), } E_4 = \frac{|P|^2}{2m_4} - \frac{k_4}{|Q|} = \frac{m_4 k_4^2}{2L_4^2}, \quad |P| \simeq |P_\parallel| \text{ and } P_\parallel < 0.$$

This gives the matrix stated in the sublemma. In part (a), all the Cartesian and Delaunay variables are immediately to the left of the section, so we have $G_4^L = O(\mu \mathcal{G}/\chi)$ using Lemma 6.9 and $x_{4,\parallel}^L \simeq \chi/2$.

Next we consider part (b). It follows from Lemma A.3 that to the right of the section $x_{4,\parallel}^L = \chi/2$ the matrix $\frac{\partial(\mathcal{V}_1; \mathcal{V}_4)^R}{\partial(\mathcal{X}_1, \mathcal{X}_4)^R}$ has the same estimates as in the left case with

$$u_{iii'} = \left(0_{1 \times 4}; 1, -\frac{L_4}{G_4^2 + L_4^2}\right)^T, \\ l_{iii'} = \frac{1}{\chi} \left(0_{1 \times 4}; \frac{\partial G_4}{\partial(x_4, v_4)}\right) = \left(0_{1 \times 4}; O\left(\frac{\mu \mathcal{G}}{\chi^2}\right), \frac{m_4 k_4}{\chi L_4}, O\left(\frac{\mu \mathcal{G}}{\chi}\right), \frac{1}{2}\right),$$

We see that the last entry of $u_{iii'}$ gets a “-” since we need to choose $\text{sign}(u) = -1$ in Lemma A.3. Moreover, $l_{iii'}$ gets a “-” sign compared to l_{iii} since both P_\parallel, Q_\parallel get “-” signs. \square

With the two sublemmas, we can complete the computation of the matrices (II) and (IV).

Computation of matrices (II) and (IV) in Proposition 5.1. To be compatible with the formal derivation in (9.1), we add four zeros to u_i as the new first four entries. We still denote the new vector of 12 components by u_i as stated in Proposition 5.1. We also define a 12×10 matrix $C = \begin{bmatrix} \text{Id}_{4 \times 4} & 0_{4 \times 6} \\ * & \end{bmatrix}$ where $*$ is the $O(1)$ matrix in Sublemma 9.1.

Then consider Sublemma 9.2. To be compatible with the formal derivation in (9.1), we enlarge $u_{iii}, u_{iii'}$ by adding four zeros as the new first four entries to get vectors in \mathbb{R}^{10} . We define a 10×12 matrix $A = \begin{bmatrix} \text{Id}_{4 \times 4} & 0_{4 \times 8} \\ 0_{6 \times 4} & * \end{bmatrix}$, where $*$ is the $O(1)$ matrix of Sublemma 9.2.

Fitting these manipulations into (9.1) gives

$$(II) = (\chi u_{iii} \otimes l_{iii} + A) R \cdot L^{-1} (\chi u_i \otimes l_i + C), \quad (IV) = (\chi u_{iii'} \otimes l_{iii'} + A) L \cdot R^{-1} (\chi u_i \otimes l_{i'} + C). \quad \square$$

10. THE LOCAL MAP

In this section, we study the local map on both \mathcal{C}^0 and \mathcal{C}^1 level. The contents of this section are a slight modification of the corresponding section of [DX] for the reason that Q_1 is so far that it does not influence the interaction of Q_3 and Q_4 very much. We present all the details of the proof here for the sake of completeness.

The section $\{|q_3 - q_4| = \mu^\kappa\}$ ($1/3 < \kappa < 1/2$) cuts the orbit for the local map into three pieces: $\{x_{4,\parallel}^R = -2\} \rightarrow \{|q_3 - q_4| = \mu^\kappa\}$, $\{|q_3 - q_4| = \mu^\kappa\} \rightarrow \{|q_3 - q_4| = \mu^\kappa\}$, and $\{|q_3 - q_4| = \mu^\kappa\} \rightarrow \{x_{4,\parallel}^R = -2\}$. We define three maps $\mathbb{L}^-, \mathbb{L}^0, \mathbb{L}^+$ corresponding to the three pieces and we have $\mathbb{L} = \mathbb{L}^+ \circ \mathbb{L}^0 \circ \mathbb{L}^-$.

10.1. \mathcal{C}^0 control of the local map. In this section, we obtain the \mathcal{C}^0 estimate of the local map, based on which we prove Lemma 2.2.

- Notation 10.1.** (1) We use the superscript $+$ (or $-$) to denote the value of the orbit parameters exiting (or entering) the sphere $|q_3 - q_4| = \mu^\kappa$.
(2) Also recall the coordinates q_-, p_- for the relative motion and q_+, p_+ for the motion of the mass center of Q_3 and Q_4 in (4.9).
(3) We introduce the notation

$$\mathbf{q} = (q_+, q_1), \quad \mathbf{p} = (p_+, p_1),$$

to handle the mass center and the remote body simultaneously.

The next lemma shows that the local map is close to elastic collision.

Lemma 10.1. (a) We have the following equations as $\mu \rightarrow 0$

$$(10.1) \quad \begin{cases} p_3^+ = \frac{1}{2} \text{Rot}(\alpha)(p_3^- - p_4^-) + \frac{1}{2}(p_3^- + p_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1}), \\ p_4^+ = -\frac{1}{2} \text{Rot}(\alpha)(p_3^- - p_4^-) + \frac{1}{2}(p_3^- + p_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1}), \\ (\mathbf{q}^+, \mathbf{p}^+) = (\mathbf{q}^-, \mathbf{p}^-) + O(\mu^\kappa), \\ |q_3^- - q_4^-| = \mu^\kappa, \quad |q_3^+ - q_4^+| = \mu^\kappa, \end{cases}$$

where $\text{Rot}(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$, and

$$(10.2) \quad \alpha = \pi - 2 \arctan \left(\frac{G_{in}}{\mu \mathcal{L}_{in}} \right), \text{ where } \frac{1}{4\mathcal{L}_{in}^2} = \frac{p_-^2}{4} - \frac{\mu}{2|q_-|}, \quad G_{in} = 2p_- \times q_-.$$

- (b) We have $\mathcal{L}_{in} = O(1)$. If α is bounded away from 0 and π by an angle independent of μ then $G_{in} = O(\mu)$ and the closest distance between q_3 and q_4 is bounded away from zero by $\delta\mu$ and from above by μ/δ for some $\delta > 0$ independent of μ .
- (c) If α is bounded away from 0 by an angle independent of μ then the angle between q_- and p_- is $O(\mu^{1-\kappa})$.
- (d) The time interval during which the orbit stays in the sphere $|q_-| = 2\mu^\kappa$ is

$$\Delta t = O(\mu^\kappa).$$

Remark 10.1. Part (d) is very intuitive. The radius of the sphere $|q_-| = 2\mu^\kappa$ is $2\mu^\kappa$. The relative velocity is $O(1)$ on the boundary of the sphere and it gets faster when q_- gets closer to 0. So the total time for the relative motion to stay inside the sphere is $O(\mu^\kappa)$.

Proof. In the proof, we omit the subscript *in* standing for the variables *inside* the sphere $|q_-| = 2\mu^\kappa$ without leading to confusion.

The idea of the proof is to treat the relative motion as a perturbation of Kepler motion and then approximate the relative velocities by their asymptotic values for the Kepler motion.

Fix a small number δ_1 . Below we derive several estimates valid for the first δ_1 units of time the orbit spends in the set $|q_-| \leq 2\mu^\kappa$. We then show that $\Delta t \ll \delta_1$. It will be convenient to measure time from the orbit enters the set $|q_-| < 2\mu^\kappa$.

Using the formula in the Appendix A.1, we decompose the last “=” of the Hamiltonian (4.10) as $H = H_{rel} + \mathfrak{h}(\mathbf{q}, \mathbf{p})$ where

$$H_{rel} = \frac{\mu^2}{4L^2} + \frac{|q_-|^2}{2|q_+|^2} - \frac{|q_+ \cdot q_-|^2}{2|q_+|^5} + O(\mu^{3\kappa}), \text{ as } 1/\chi \ll \mu \rightarrow 0,$$

and \mathfrak{h} depends only on \mathbf{q} and \mathbf{p} .

Note that H is preserved and $\dot{\mathfrak{h}} = O(1)$ which implies that $\frac{L}{\mu}$ is $O(1)$ and moreover

that ratio does not change much for $t \in [0, \delta_1]$. Using the identity $\frac{\mu^2}{4L^2} = \frac{p_-^2}{4} - \frac{\mu}{2|q_-|}$,

$L = \mu\mathcal{L}$, we see that initially $\frac{L}{\mu}$ is uniformly bounded from below for the orbits from Lemma 2.2. Thus there is a constant δ_2 such that for $t \in [0, \delta_1]$ we have $\delta_2\mu \leq L(t) \leq \frac{\mu}{\delta_2}$.

Expressing the Cartesian variables via Delaunay variables (c.f. equation (A.3) in Section A.2) we have up to a rotation by g

$$(10.3) \quad \begin{aligned} q_{\parallel} &= \frac{1}{\mu} L^2 (\cosh u - e), & q_{\perp} &= \frac{1}{\mu} L G \sinh u, \\ O(\mu^\kappa) = |q_-| &= \frac{L^2}{\mu} (e \cosh u - 1), \end{aligned}$$

using (6.15), where $u - e \sinh u = \ell$. This gives $\ell = O(\mu^{\kappa-1})$.

Next

$$\dot{\ell} = -\frac{\partial H}{\partial L} = -\frac{\mu^2}{2L^3} - \frac{\partial H_{rel}}{\partial q_-} \frac{\partial q_-}{\partial L} = -\frac{\mu^2}{2L^3} + O(\mu^\kappa)O(\mu^{\kappa-1}) = -\frac{\mu^2}{2L^3} + O(\mu^{2\kappa-1}).$$

Since the leading term here is at least $\frac{\delta_2^3}{2\mu}$ while $\ell = O(\mu^{\kappa-1})$ we obtain part (d) of the lemma. In particular the estimates derived above are valid for the time the orbits spend in $|q_-| \leq 2\mu^\kappa$. Next, without using any control on G (using the inequality $\left| \frac{\partial e}{\partial G} \right| = \frac{1}{L} \frac{G/L}{e} \leq \frac{1}{L}$), we have

$$(10.4) \quad \dot{G} = \frac{\partial H}{\partial q_-} \frac{\partial q_-}{\partial g} = O(|q_-|^2) = O(\mu^{2\kappa}), \quad \dot{L} = \frac{\partial H}{\partial q_-} \frac{\partial q_-}{\partial \ell} = O(\mu^{\kappa+1}),$$

$$(10.5) \quad \dot{g} = \frac{\partial H}{\partial q_-} \frac{\partial q_-}{\partial G} = O(\mu^\kappa)O(\mu^{\kappa-1}) = O(\mu^{2\kappa-1}).$$

Integrating over time $\Delta t = O(\mu^\kappa)$ we get the oscillation of g and $\arctan \frac{G}{L}$ are $O(\mu^{3\kappa-1})$.

We are now ready to derive the first two equations of (10.1). It is enough to show $p_-^+ = \text{Rot}(\alpha)p_-^- + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1})$ where $\alpha = \pi - 2 \arctan \frac{G}{L}$ is the angle formed by the two asymptotes of the Kepler hyperbolic motion. We first have $|p_-^+| = |p_-^-| + O(\mu^\kappa)$ using the total energy conservation. It remains to show the expression of α . Let us denote till the end of the proof $\phi = \arctan \frac{G}{L}, \gamma = \frac{(1/2) - \kappa}{3}$. Recall (see (A.3)) that for $p_- = (p_\parallel, p_\perp)$,

$$(10.6) \quad p_\parallel = \tilde{p}_\parallel \cos g + \tilde{p}_\perp \sin g, \quad p_\perp = -\tilde{p}_\parallel \sin g + \tilde{p}_\perp \cos g, \quad \text{where}$$

$$\tilde{p}_\parallel = \frac{\mu}{L} \frac{\sinh u}{1 - e \cosh u}, \quad \tilde{p}_\perp = \frac{\mu G}{L^2} \frac{\cosh u}{1 - e \cosh u}.$$

Consider two cases.

(I) $G \leq \mu^{\kappa+\gamma}$. In this case on the boundary of the sphere $|q_-| = 2\mu^\kappa$ we have $\ell > \delta_3 \mu^{-\gamma}$ for some constant δ_3 . Thus

$$\frac{p_\perp}{p_\parallel} = \frac{\frac{\mu G}{L^2} \cosh u \cos g + \frac{\mu}{L} \sinh u \sin g}{-\frac{\mu G}{L^2} \cosh u \sin g + \frac{\mu}{L} \sinh u \cos g} = \frac{\frac{G}{L} \pm \tan g}{\pm 1 - \frac{G}{L} \tan g} + O(e^{-2|u|}) = \tan(g \pm \phi) + O(\mu^{2\gamma}).$$

where the plus sign is taken if $u > 0$ and the minus sign is taken if $u < 0$. The angle formed by the two asymptotes can either be $2 \arctan \frac{G}{L}$ or $\pi - 2 \arctan \frac{G}{L}$. To decide which to choose as α , we notice $\alpha = \pi, G = 0$ correspond to the collisional and bouncing back orbit, and $\alpha = 0, G = \infty$ correspond to free motion.

(II) $G > \mu^{\kappa+\gamma}$. In this case $\frac{G}{L} \gg 1$ and so it suffices to show that $\frac{p_\perp}{p_\parallel}$ (or $\frac{p_\parallel}{p_\perp}$) changes little during the time the orbit is inside the sphere. Consider first the case where $|g^-| > \frac{\pi}{4}$ so $\sin g$ is bounded from below. Then

$$\frac{p_\perp}{p_\parallel} = \cot g + O(\mu^{1-(\kappa+\gamma)})$$

proving the claim of part (a) in that case. The case $|g^-| \leq \frac{\pi}{4}$ is similar but we need to consider $\frac{p_{\parallel}}{p_{\perp}}$. This completes the proof in case (II).

Integrating over time $\Delta t = O(\mu^\kappa)$ the (\mathbf{q}, \mathbf{p}) equations obtained from Hamiltonian (4.10), we obtain

$$(10.7) \quad (\mathbf{q}, \mathbf{p})^+ = (\mathbf{q}, \mathbf{p})^- + O(\mu^\kappa).$$

We also have $(p, q)_-^+ = q_-^- + O(\mu^\kappa)$ due to the definition of the sections $\{|q_-^\pm| = 2\mu^\kappa\}$. This completes the proof of part (a).

The first claim of part (b) has already been established. The estimate of G follows from the formula for α . The estimate of the closest distance follows from the fact that if α is bounded away from 0 and π then the q_- orbit of $q_-(t)$ is a small perturbation of Kepler motion. For Kepler motion the closest distance is related

to G as follows $ae - a = a \left(\sqrt{1 + \frac{G^2}{L^2}} - 1 \right) = \frac{G^2/\mu}{e+1}$, where $a = L^2/\mu$ is the semi-

major (See Appendix A.2). We integrate the \dot{G} equation (10.4) over time $O(\mu^\kappa)$ to get the total variation ΔG is at most $\mu^{3\kappa}$, which is much smaller than μ . So G is bounded away from 0 by a quantity of order $O(\mu)$.

Finally part (c) follows since we know $G = 2\mu^\kappa |p_-| \sin \angle(p_-, q_-) = O(\mu)$. \square

10.2. Approaching the close encounter. To study the \mathcal{C}^1 estimate of the local map, we first show that \mathbb{L}^+ and \mathbb{L}^- are negligible and we then focus on \mathbb{L}^0 .

Lemma 10.2. *Consider the maps \mathbb{L}^\pm under the assumption of Lemma 3.1. Then the vectors $\bar{\mathbf{l}}_j, \bar{\bar{\mathbf{l}}}_j$, $j = 1, 2$, are almost left invariant by $d\mathbb{L}^+$ and $\text{span}\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\}$ is almost right invariant by $d\mathbb{L}^-$ in the following sense:*

$$d\mathbb{L}^- \cdot \text{span}\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\} = \text{span}\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\} + o(1),$$

$$\bar{\mathbf{l}}_j \cdot d\mathbb{L}^+ = \bar{\mathbf{l}}_j + o(1), \quad \bar{\bar{\mathbf{l}}}_j \cdot d\mathbb{L}^+ = \bar{\bar{\mathbf{l}}}_j + o(1),$$

as $1/\chi \ll \mu \rightarrow 0$.

Proof. Step 0. We first establish that the matrices $d\mathbb{L}^\pm$ are of the forms:

$$d\mathbb{L}^- = (\text{Id}_{10} + u_5^f \otimes (O(1)_{1 \times 10}))(\text{Id}_{10} + o(1))(\text{Id}_{10} + u_5^f \otimes l_5^f),$$

$$d\mathbb{L}^+ = (\text{Id}_{10} + u_1^i \otimes l_1^i)(\text{Id}_{10} + o(1))(\text{Id}_{10} + u_1^i \otimes (O(1)_{1 \times 10})).$$

Once this is done, it is straightforward to check the statement in the lemma using the explicit expression of $d\mathbb{L}^\pm$ and $\bar{\mathbf{l}}, \bar{\bar{\mathbf{l}}}, \bar{\mathbf{u}}, \bar{\bar{\mathbf{u}}}$ since we have $u_1^i \sim u_5^f \rightarrow (0, 1, 0_{1 \times 8})^T = \tilde{w}$ and $l_1^i \sim l_5^f \rightarrow (1, 0_{1 \times 9}) = \hat{\bar{\mathbf{l}}}$ as $1/\chi \ll \mu \rightarrow 0$, where $\tilde{w}, \hat{\bar{\mathbf{l}}}$ are defined in the statement of Lemma 3.2.

Again we use formula (8.1) to reduce the calculation to the boundary terms and the solution of the variational equation.

Step 1. The integral of variational equations has the form of $\text{Id}_{10} + o(1)$. The reason is, we need to integrate the variational equations (the same estimates as the right case of the two matrices of Lemma 7.2) over time $O(1)$ then add to identity. Moreover, when Q_3, Q_4 come to close encounter, we only need to worry about the term $\frac{\mu}{|x_4 - \frac{x_3}{1+\mu}|} = \frac{\mu}{|q_4 - q_3|}$ in Hamiltonian (4.7) whose contribution to the

variational equation is $\frac{\mu}{|q_4 - q_3|^3}$ and to the solution of the variational equation has the order

$$(10.8) \quad O\left(\int_{-2}^{\mu^\kappa} \frac{\mu}{|t|^3} dt\right) = O(\mu^{1-2\kappa}) \ll 1.$$

Similar consideration shows that the perturbation from x_1, v_1 is $O(1/\chi^3)$.

On the other hand absence of perturbation, all Delaunay variables except ℓ_3 are constants of motion. The $(2, 1)$ entry is also $o(1)$ following from the same estimate as the $(2, 1)$ entry of the matrix in Lemma 7.2. After integrating over time $O(1)$, the solutions to the variational equations have the form

$$\text{Id} + O(\mu^{1-2\kappa} + 1/\chi^3).$$

Step 2. To study the boundary contributions, it is enough to work out $\mathcal{F} \otimes \frac{\partial \ell_4}{\partial \mathcal{V}}$ using (8.1). For the boundary $x_{4,\parallel}^R = -2$, we get the estimate the same as Section 8, namely,

$$\mathcal{F} = u_5^f \sim u_1^i = (0, 1, 0_{1 \times 8}) + o(1), \quad \frac{\partial \ell_4}{\partial \mathcal{V}} = l_5^f \sim l_1^i = (1, 0_{1 \times 9}) + o(1).$$

For the section $\{|q_3 - q_4| = \mu^\kappa\}$, we first have from Section 8

$$\mathcal{F} \sim u_1^i \sim u_5^f = (0, 1, 0_{1 \times 8}) + o(1).$$

Next we work on $\frac{\partial \ell_4}{\partial \mathcal{V}}$. We have

$$(10.9) \quad \frac{\partial \ell_4}{\partial \mathcal{V}} = -\frac{\frac{\partial |q_3 - q_4|}{\partial \mathcal{V}}}{\frac{\partial |q_3 - q_4|}{\partial \ell_4}} = -\frac{(q_3 - q_4) \cdot \frac{\partial (q_3 - q_4)}{\partial \mathcal{V}}}{(q_3 - q_4) \cdot \frac{\partial (q_3 - q_4)}{\partial \ell_4}} = \frac{(p_3 - p_4) \cdot \frac{\partial (q_3 - q_4)}{\partial \mathcal{V}}}{(p_3 - p_4) \cdot \frac{\partial q_4}{\partial \ell_4}} + O(\mu^{1-\kappa})$$

where in the last “=” we use Lemma 10.1(c) that the angle formed by $q_3 - q_4$ and $p_3 - p_4$ is $O(\mu^{1-\kappa})$ to replace $q_3 - q_4$ by $p_3 - p_4$ making $O(\mu^{1-\kappa})$ error. Note that $\frac{\partial q_4}{\partial \ell_4}$ is parallel to p_4 . Using the information about v_3 and v_4 from Appendix B.1 we see that $\langle p_3, p_4 \rangle \neq \langle p_4, p_4 \rangle$. Therefore the denominator in (10.9) is bounded away from zero and so

$$\frac{\partial \ell_4}{\partial \mathcal{V}} = (O(1)_{1 \times 10}).$$

We also need to make sure the second component $\frac{\partial \ell_4}{\partial \ell_3}$ is not close to 1, so that

$\text{Id} - \mathcal{F}(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial \mathcal{V}^f}$ is invertible when $|q_3 - q_4| = \mu^\kappa$ serves as the final section. In fact, we have from (6.3) that $\frac{\partial \ell_4}{\partial \ell_3} \simeq -1$. Using formula (8.1), we get the asymptotics in Step 0. The proof is now complete. \square

10.3. \mathcal{C}^1 control of the local map, proof of Lemma 3.1. Here we give the proof of Lemma 3.1. Our goal is to show that the main contribution to the derivative comes from differentiating the main term in Lemma 10.1.

Proof of Lemma 3.1. Step 0, Coordinates changes outside the sphere.

We first convert Delaunay variables to Cartesian variables. The derivative is

$$\frac{\partial \mathcal{V}}{\partial \mathcal{X}} = \text{diag} \left\{ \frac{\partial(L, \ell, G, g)_3}{\partial(x, v)_3}, \text{Id}_4, \frac{\partial(G, g)_4}{\partial(x, v)_4} \right\} = O(1).$$

Then we go from $(x_3, v_3; x_1, v_1; x_4, v_4)$ to $(q_3, p_3; q_1, p_1; q_4, p_4)$. The derivative is $\text{Id}_{12} + O(\mu)$ using (2.3). The next step is to go from $(q_3, p_3; q_1, p_1; q_4, p_4)$ to

$$(q_-, p_-; q_1, p_1; q_+, p_+). \text{ The derivative is } \begin{bmatrix} -\frac{1}{2}\text{Id}_2 & 0 & 0 & \frac{1}{2}\text{Id}_2 & 0 \\ 0 & -\text{Id}_2 & 0 & 0 & \text{Id}_2 \\ 0 & 0 & \text{Id}_4 & 0 & 0 \\ \frac{1}{2}\text{Id}_2 & 0 & 0 & \frac{1}{2}\text{Id}_2 & 0 \\ 0 & \text{Id}_2 & 0 & 0 & \text{Id}_2 \end{bmatrix}. \text{ All}$$

three matrices are $O(1)$. So we reduce the problem to proving the structure of $d\mathbb{L}$ in the lemma for $\frac{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^+}{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^-}$.

As before we use the formula (8.1). We need to consider the integration of the variational equations and also the boundary contribution. We need to use the following bounds

$$\mathcal{L} \sim 1, \quad G_- \sim \mu, \quad \ell_- \sim \mu^{\kappa-1}, \quad |q_-| \leq \mu^\kappa, \quad |q_1| \sim \chi, \quad |q_+|, |p_1|, |p_+| \sim 1.$$

Step 1, the Hamiltonian equations, the variational equations and the boundary contributions.

It is convenient to use the variable $\mathcal{L} = L/\mu$. From the Hamiltonian (4.9), we have $\dot{\ell} = -\frac{1}{2\mu\mathcal{L}^3} + O(\mu^{2\kappa})$. Using ℓ as the time variable we get from (4.9) that the equations for relative motion take the following form (recall that the scale for ℓ is $O(\mu^{\kappa-1})$):

$$(10.10) \quad \begin{cases} \frac{\partial \mathcal{L}}{\partial \ell} = -2\mathcal{L}^3 \frac{\partial H}{\partial \ell} \left(1 - 2\mathcal{L}^3 \frac{\partial H}{\partial \mathcal{L}} + \dots \right) (1 + O(\mu^{2\kappa+1})) = O(\mu^{1+\kappa}), \\ \frac{\partial G}{\partial \ell} = -2\mu\mathcal{L}^3 \frac{\partial H}{\partial g} \left(1 - 2\mathcal{L}^3 \frac{\partial H}{\partial \mathcal{L}} + \dots \right) (1 + O(\mu^{2\kappa+1})) = O(\mu^{1+2\kappa}), \\ \frac{\partial g}{\partial \ell} = 2\mu\mathcal{L}^3 \frac{\partial H}{\partial G} \left(1 - 2\mathcal{L}^3 \frac{\partial H}{\partial \mathcal{L}} + \dots \right) (1 + O(\mu^{2\kappa+1})) = O(\mu^{2\kappa}), \\ \frac{d\mathbf{q}}{d\ell} = -2\mu\mathcal{L}^3 \left[\begin{array}{c} \frac{p_+}{2} + \mu(p_+ + p_1) \\ 2\mu p_1 + \mu p_+ \end{array} \right] (1 + O(\mu^{2\kappa+1})) = O(\mu), \\ \frac{d\mathbf{p}}{d\ell} = 2\mu\mathcal{L}^3 \left[\begin{array}{c} \frac{q_+}{|q_+|^3} + O(\mu^{2\kappa}) \\ \frac{1+2\mu}{\mu} \frac{q_1}{|q_1|^3} + O(1/\chi^3) \end{array} \right] (1 + O(\mu^{2\kappa+1})) = O(\mu). \end{cases}$$

where \dots denote the lower order terms. The estimates of the second and third equations follow from (10.4) and (10.5) while the estimate of the other three equations is similar. In the first three equations, the main contribution in H is coming from $|q_-|^2$ and $|q_+ \cdot q_-|^2$. In the $\frac{d\mathbf{q}}{d\ell}, \frac{d\mathbf{p}}{d\ell}$ equations, the main contribution is given by $\frac{dq_+}{d\ell}, \frac{dp_+}{d\ell}$, so we are in the situation of [DX]. The following proofs are almost the same as that in [DX].

Next we analyze the variational equations. This estimate is much easier than that of the global map part.

$$(10.11) \quad \frac{d}{d\ell} \begin{bmatrix} \delta\mathcal{L} \\ \delta G \\ \delta g \\ \delta\mathbf{q} \\ \delta\mathbf{p} \end{bmatrix} = O \begin{pmatrix} \mu^{1+\kappa} & \mu^\kappa & \mu^{2\kappa} & \mu^{1+\kappa} & 0 \\ \mu^{2+\kappa} & \mu^{2\kappa} & \mu^{1+2\kappa} & \mu^{1+2\kappa} & 0 \\ \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} & \mu^{2\kappa} & 0 \\ \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2} & \mu^{2\kappa+2} & \mu \\ \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2} & \mu & 0 \end{pmatrix} \begin{bmatrix} \delta\mathcal{L} \\ \delta G \\ \delta g \\ \delta\mathbf{q} \\ \delta\mathbf{p} \end{bmatrix}.$$

Integrating this equation over time $\mu^{\kappa-1}$. It is enough to use two steps of Picard iteration to find fundamental solution of the variational equation is

$$(10.12) \quad \text{Id} + O \left(\begin{array}{ccc|cc} \mu^{6\kappa-2} & \mu^{6\kappa-3} & \mu^{3\kappa-1} & \mu^{6\kappa-2} & \mu^{3\kappa} \\ \mu^{6\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa} & \mu^{3\kappa} & \mu^{4\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} & \mu^{3\kappa-1} & \mu^{4\kappa-1} \\ \hline \mu^\kappa & \mu^{3\kappa-1} & \mu^{4\kappa-1} & \mu^{2\kappa} & \mu^\kappa \\ \mu^\kappa & \mu^{3\kappa-1} & \mu^{4\kappa-1} & \mu^\kappa & \mu^{2\kappa} \end{array} \right).$$

This calculation can either be done by hand or use computer.

Next, we compute the boundary contribution. In terms of the Delaunay variables inside the sphere $|q_-| = 2\mu^\kappa$, we have

$$(10.13) \quad \frac{\partial\ell}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})} = - \left(\frac{\partial|q_-|}{\partial\ell} \right)^{-1} \frac{\partial|q_-|}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})} = (O(\mu^{\kappa-1}), O(\mu^{\kappa-2}), 0; 0_{1 \times 4}, 0_{1 \times 4}).$$

Indeed, due to (10.3) we have $\frac{\partial|q_-|}{\partial g} = 0$, $\frac{\partial|q_-|}{\partial\ell} = O(\mu)$, $\frac{\partial|q_-|}{\partial\mathcal{L}} = O(\mu^\kappa)$ and $\frac{\partial|q_-|}{\partial G} = O(\mu^{\kappa-1})$. Combining this with (10.10) we get

$$(10.14) \quad \left(\frac{\partial\mathcal{L}}{\partial\ell}, \frac{\partial G}{\partial\ell}, \frac{\partial g}{\partial\ell}, \frac{\partial\mathbf{q}}{\partial\ell}, \frac{\partial\mathbf{p}}{\partial\ell} \right) \otimes \frac{\partial\ell}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})} = O(\mu^{1+\kappa}, \mu^{1+2\kappa}, \mu^{2\kappa}; \mu_{1 \times 4}, \mu_{1 \times 4}) \otimes O(\mu^{\kappa-1}, \mu^{\kappa-2}, 0; 0_{1 \times 4}, 0_{1 \times 4}).$$

Step 2, the analysis of the relative motion part.

The tensor part of $d\mathbb{L}$ comes mainly from the relative motion part, so in this step focus only on the relative motion part. The perturbation from \mathbf{p}, \mathbf{q} will be studied in the next step.

Using (8.1) we obtain the derivative matrix

$$(10.15) \quad \begin{aligned} \frac{\partial(\mathcal{L}, G, g)^+}{\partial(\mathcal{L}, G, g)^-} &= \left(\text{Id} + O \begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{pmatrix} \right)^{-1} \times \\ &\quad \left(\text{Id} + O \begin{pmatrix} \mu^{6\kappa-2} & \mu^{6\kappa-3} & \mu^{3\kappa-1} \\ \mu^{6\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix} \right) \left(\text{Id} - O \begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{pmatrix} \right) \\ &= \text{Id} + O \begin{pmatrix} \mu^{6\kappa-2} & \mu^{6\kappa-3} & \mu^{3\kappa-1} \\ \mu^{6\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix} := \text{Id} + P. \end{aligned}$$

We are now ready to compute the relative motion part of the derivative of the Poincaré map. For q_- , we are only interested in the angle $\Theta := \arctan \left(\frac{q_{-, \perp}}{q_{-, \parallel}} \right)$ since the length $|(q_{-, \perp}, q_{-, \parallel})| = 2\mu^\kappa$ is fixed when restricted on the sphere.

We split the derivative matrix as follows:

$$(10.16) \quad \frac{\partial(\Theta_-, p_-)^+}{\partial(\Theta_-, p_-)^-} = \frac{\partial(\Theta_-, p_-)^+}{\partial(\mathcal{L}, G, g)^+} \frac{\partial(\mathcal{L}, G, g)^+}{\partial(\mathcal{L}, G, g)^-} \frac{\partial(\mathcal{L}, G, g)^-}{\partial(\Theta_-, p_-)^-} = \\ \frac{\partial(\Theta_-, p_-)^+}{\partial(\mathcal{L}, G, g)^+} \frac{\partial(\mathcal{L}, G, g)^-}{\partial(\Theta_-, p_-)^-} + \frac{\partial(\Theta_-, p_-)^+}{\partial(\mathcal{L}, G, g)^+} P \frac{\partial(\mathcal{L}, G, g)^-}{\partial(\Theta_-, p_-)^-} = I + II.$$

Using equations (10.3) and (10.6) we obtain

$$(10.17) \quad \frac{\partial(\Theta_-, p_-)^+}{\partial(\mathcal{L}, G, g)^+} = O \begin{pmatrix} 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \end{pmatrix}.$$

Next, we consider the first term in (10.16).

$$(10.18) \quad I = \frac{\partial(\Theta_-, p_-)^+}{\partial \mathcal{L}^+} \otimes \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-} + \frac{\partial(\Theta_-, p_-)^+}{\partial G^+} \otimes \frac{\partial G^-}{\partial(\Theta_-, p_-)^-} + \frac{\partial(\Theta_-, p_-)^+}{\partial g^+} \otimes \frac{\partial g^-}{\partial(\Theta_-, p_-)^-}.$$

Using the expressions

$$\frac{1}{4\mathcal{L}^2} = \frac{p_-^2}{4} - \frac{\mu}{2|q_-|}, \quad G = p_- \times q_- = |p_-| \cdot |q_-| \sin \angle(p_-, q_-)$$

we see that

$$(10.19) \quad \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-} = O(1), \quad \frac{\partial G^-}{\partial(\Theta_-, p_-)^-} = (O(\mu^\kappa), O(\mu^\kappa)).$$

Next, we have $\frac{\partial(\Theta_-, p_-)^+}{\partial g^+} = (O(1), O(1))$ from equations (10.3) and (10.6). To obtain the derivatives of g we use the fact that

$$\frac{p_{-, \perp}}{p_{-, \parallel}} = \frac{\sin g \sinh u \pm \frac{G}{\mu \mathcal{L}} \cos g \cosh u}{\cos g \sinh u \mp \frac{G}{\mu \mathcal{L}} \sin g \cosh u} = \frac{\tan g \pm \frac{G}{\mu \mathcal{L}}}{1 \mp \frac{G}{\mu \mathcal{L}} \tan g} + e^{-2|u|} E(G/\mu \mathcal{L}, g, u),$$

where E is a smooth function satisfying $\frac{\partial E}{\partial g} = O(1)$ as $\ell \rightarrow \infty$. Therefore we get

$$g = \arctan \left(\frac{p_{-, \perp}}{p_{-, \parallel}} - e^{-2|u|} E(G/\mu \mathcal{L}, g, u) \right) \mp \arctan \frac{G}{\mu \mathcal{L}} \pmod{\pi}, \quad \text{as } \ell \rightarrow \infty.$$

We choose the $+$ when considering the incoming orbit parameters. Thus

$$\frac{\partial g}{\partial(\Theta_-, p_-)} \left(1 + O(e^{-2|u|}) \right) = \frac{\partial \arctan \frac{p_{-, \perp}}{p_{-, \parallel}}}{\partial(\Theta_-, p_-)} + \frac{\partial \arctan \frac{G}{\mu \mathcal{L}}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial(\Theta_-, p_-)} \\ + \left(\frac{\partial \arctan \frac{G}{\mu \mathcal{L}}}{\partial G} + O(e^{-2|u|}/\mu) \right) \frac{\partial G}{\partial(\Theta_-, p_-)} + O(e^{-2|u|}).$$

Hence

$$(10.20) \quad \frac{\partial g}{\partial(\Theta_-, p_-)} = O \left(\frac{1}{\mu} \right) \frac{\partial G}{\partial(\Theta_-, p_-)} + O(1),$$

where the $1/\mu$ comes from $\frac{\partial \arctan \frac{G^-}{\mu \mathcal{L}^-}}{\partial G^-}$ and all other terms are $O(1)$ or even smaller. Therefore

$$(10.21) \quad I = \frac{1}{\mu} \left(\mu \frac{\partial(\Theta_-, p_-)^+}{\partial G^+} + \mu \frac{\partial \arctan \frac{G^-}{\mu \mathcal{L}^-}}{\partial G^-} \frac{\partial(\Theta_-, p_-)^+}{\partial g^+} + O(e^{-2|u|}) \right) \otimes \frac{\partial G^-}{\partial(\Theta_-, p_-)^-} +$$

$$\left(\frac{\partial(\Theta_-, p_-)^+}{\partial \mathcal{L}^+} \otimes \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-} + \frac{\partial(\Theta_-, p_-)^+}{\partial g^+} \otimes \left(\frac{\partial \arctan \frac{p_-^-, \perp}{p_-^-, \parallel}}{\partial(\Theta_-, p_-)^-} + \frac{\partial \arctan \frac{G^-}{\mu \mathcal{L}^-}}{\partial \mathcal{L}^-} \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-} \right) \right)$$

$$+ O(e^{-2|u|}). \text{ Since the expression in parenthesis of the first term is } O(1), I \text{ has the}$$

rate of growth required in Lemma 3.1.

Now we study the second term in (10.16)

$$(10.22) \quad II = O \begin{pmatrix} 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \end{pmatrix} \cdot O \begin{pmatrix} \mu^{6\kappa-2} & \mu^{6\kappa-3} & \mu^{3\kappa-1} \\ \mu^{6\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix} \frac{\partial(\mathcal{L}, G, g)^-}{\partial(\Theta_-, p_-)^-}$$

$$= O \begin{pmatrix} \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix} \frac{\partial(\mathcal{L}, G, g)^-}{\partial(\Theta_-, p_-)^-}$$

$$= O \begin{pmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{pmatrix} \otimes \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-}$$

$$+ O \begin{pmatrix} \mu^{3\kappa-2} \\ \mu^{3\kappa-2} \\ \mu^{3\kappa-2} \end{pmatrix} \otimes \frac{\partial G^-}{\partial(\Theta_-, p_-)^-} + O \begin{pmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{pmatrix} \otimes \frac{\partial g^-}{\partial(\Theta_-, p_-)^-}$$

where we use that $\mu^{2\kappa} < \mu^{3\kappa-1}$ and $\mu^{2\kappa-1} < \mu^{3\kappa-2}$ since $\kappa < 1/2$. The first summand in (10.22) is $O(\mu^{3\kappa-1})$. Therefore (10.20) implies that

$$(10.23) \quad II = \frac{1}{\mu} O \begin{pmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{pmatrix} \otimes \frac{\partial G^-}{\partial(\Theta_-, p_-)^-} + O(\mu^{3\kappa-1}).$$

Now we combine (10.21) and (10.23) to get

$$(10.24) \quad \frac{\partial(\Theta_-, p_-)^+}{\partial(\Theta_-, p_-)^-} = \frac{1}{\mu} \left(\mu \frac{\partial(\Theta_-, p_-)^+}{\partial G^+} + \mu \frac{\partial \arctan \frac{G^-}{\mu \mathcal{L}^-}}{\partial G^-} \frac{\partial(\Theta_-, p_-)^+}{\partial g^+} + O(\mu^{3\kappa-1}) \right)$$

$$\otimes \frac{\partial G^-}{\partial(\Theta_-, p_-)^-} + \left(\frac{\partial(\Theta_-, p_-)^+}{\partial \mathcal{L}^+} \otimes \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-} \right.$$

$$\left. + \frac{\partial(\Theta_-, p_-)^+}{\partial g^+} \otimes \left(\frac{\partial \arctan \frac{p_-^-, \perp}{p_-^-, \parallel}}{\partial(\Theta_-, p_-)^-} + \frac{\partial \arctan \frac{G^-}{\mu \mathcal{L}^-}}{\partial \mathcal{L}^-} \frac{\partial \mathcal{L}^-}{\partial(\Theta_-, p_-)^-} \right) + O(\mu^{3\kappa-1}) \right).$$

(10.24) has the structure stated in the lemma. In (10.24), we use the variable Θ_- for the relative position q_- and we have $\frac{\partial G^-}{\partial(\Theta_-, p_-)^-} = O(\mu^\kappa)$. To get back to q_- , i.e.

to obtain $\frac{\partial(q_-, p_-)^+}{\partial(q_-, p_-)^-}$, we use $q_- = 2\mu^\kappa(\cos \Theta_-, \sin \Theta_-)$. So we have the estimate $\frac{\partial q_-^+}{\partial(\mathcal{L}_-, G_-, g_-)^+} = O(\mu^\kappa) \frac{\partial \Theta_-^+}{\partial(\mathcal{L}_-, G_-, g_-)^+} = O(\mu^{\kappa-1})$. To get $\frac{\partial -}{\partial q_-^-}$, we use the transformation from polar coordinates to Cartesian, $\frac{\partial -}{\partial q_-^-} = \frac{\partial -}{\partial(r_-, \Theta_-)^-} \frac{\partial(r_-, \Theta_-)^-}{\partial q_-^-}$, where $r_- = |q_-| = 2\mu^\kappa$. Therefore we have

$$\frac{\partial r_-^-}{\partial q_-^-} = 0, \quad \frac{\partial -}{\partial q_-^-} = \frac{1}{2\mu^\kappa} \frac{\partial -}{\partial \Theta_-^-} (-\sin \Theta_-^-, \cos \Theta_-^-).$$

So we have the estimate $\frac{\partial G_-^-}{\partial q_-^-} = O(1)$, and $\frac{\partial \mathcal{L}_-^-}{\partial q_-^-} = \frac{\partial \mathcal{L}_-^-}{\partial \Theta_-^-} = 0$ since in the expression

$\frac{1}{4\mathcal{L}^2} = \frac{p_-^2}{4} - \frac{\mu}{2|q_-|}$, the angle Θ_- plays no role. Finally, we have $\frac{\partial \arctan \frac{p_2^-}{p_1^-}}{\partial q_-^-} = 0$. So we get

$$\frac{\partial(q_-, p_-)^+}{\partial(q_-, p_-)^-} = \frac{1}{\mu} (O(\mu^\kappa)_{1 \times 2}, O(1)_{1 \times 2}) \otimes (O(1)_{1 \times 2}, O(\mu^\kappa)_{1 \times 2}) + O(1)_{4 \times 4} + O(\mu^{3\kappa-1}).$$

It remains to control the contribution coming from (\mathbf{q}, \mathbf{p}) .

Step 3, completing the proof, the contribution from the motion of the mass center.

Consider the following decomposition using (10.14)

$$\begin{aligned} (10.25) \quad & \frac{\partial(q_-, p_-; \mathbf{q}, \mathbf{p})^+}{\partial(q_-, p_-; \mathbf{q}, \mathbf{p})^-} = \frac{\partial(q_-, p_-; \mathbf{q}, \mathbf{p})^+}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^+} \frac{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^+}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^{(\ell^f)}} \\ & \frac{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^{(\ell^f)}}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^{(\ell^i)}} \frac{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^{(\ell^i)}}{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^-} \frac{\partial(\mathcal{L}, G, g; \mathbf{q}, \mathbf{p})^-}{\partial(q_-, p_-; \mathbf{q}, \mathbf{p})^-}. \\ & := \begin{bmatrix} M & 0 \\ 0 & \text{Id} \end{bmatrix} \begin{bmatrix} A & 0 \\ B & \text{Id} \end{bmatrix} \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} A' & 0 \\ B' & \text{Id} \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & \text{Id} \end{bmatrix} \\ & = \begin{bmatrix} MACA'N + MADB'N & MAD \\ (BC + E)A'N + (BD + F)B'N & BD + F \end{bmatrix} \end{aligned}$$

We already know all the matrices above. See (10.12) for C, D, E, F , (10.14) for A, B, A', B' , and (10.17), (10.19), (10.20) for $M = O \begin{pmatrix} \mu^\kappa & \mu^{\kappa-1} & \mu^\kappa \\ 1 & \mu^{-1} & 1 \end{pmatrix}$, and N . Moreover, $ACA' = \text{Id} + P$ is given by (10.15). It is a straightforward computation that CA' dominates DB' , so ADB' provides a small correction to the P in $ACA' = \text{Id} + P$ in (10.15). Therefore $MACA'N + MADB'N$ in (10.25) has the same structure as $MACA'N$ obtained in (10.21) and (10.22). Next we have

$$\begin{aligned}
BD + F &= O(\mu, \mu) \otimes O(\mu^{\kappa-1}, \mu^{\kappa-2}, 0) O \begin{pmatrix} \mu^{6\kappa-2} & \mu^{3\kappa} \\ \mu^{3\kappa} & \mu^{4\kappa} \\ \mu^{3\kappa-1} & \mu^{4\kappa-1} \end{pmatrix} + \text{Id} + \begin{pmatrix} \mu^{2\kappa} & \mu^{\kappa} \\ \mu^{\kappa} & \mu^{2\kappa} \end{pmatrix} \\
&= \text{Id} + O \begin{pmatrix} \mu^{\kappa} & \mu^{\kappa} \\ \mu^{\kappa} & \mu^{2\kappa} \end{pmatrix}. \\
BC + E &= O(\mu, \mu) \otimes O(\mu^{\kappa-1}, \mu^{\kappa-2}, 0) O \begin{pmatrix} \mu^{6\kappa-2} & \mu^{6\kappa-3} & \mu^{3\kappa-1} \\ \mu^{6\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix} \\
&+ \begin{pmatrix} \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2} \\ \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2} \end{pmatrix} = O \begin{pmatrix} \mu^{7\kappa-2} & \mu^{4\kappa-2} & \mu^{4\kappa-1} \\ \mu^{7\kappa-2} & \mu^{4\kappa-2} & \mu^{4\kappa-1} \end{pmatrix}.
\end{aligned}$$

Accordingly

$$(10.26) \quad (BC + E)A'N + (BD + F)B'N = \frac{1}{\mu} [O(\mu^{\kappa})]_{1 \times 4} \otimes \frac{\partial G^-}{\partial(q, p)_-} + O(\mu^{\kappa}).$$

Finally, we have $MAD = [O(\mu^{3\kappa-1})]_{3 \times 2}$.

These estimates of the matrix (10.25) are enough to conclude the Lemma. To summarize, we get the resulting derivative estimate as

$$(10.27) \quad (10.25) = \frac{1}{\mu} O(\mu_{1 \times 2}^{\kappa}, 1_{1 \times 2}, \mu_{1 \times 8}^{\kappa}) \otimes O(1_{1 \times 2}, \mu_{1 \times 2}^{\kappa}, 0_{1 \times 8}) + \begin{bmatrix} (O(1))_{4 \times 4} & O(\mu^{3\kappa-1}) \\ O(\mu^{\kappa}) & \text{Id}_8 + O(\mu^{\kappa}) \end{bmatrix},$$

□

The above proof actually gives us more information. Below we use the Delaunay variables $(L_3, \ell_3, G_3, g_3, G_4, g_4)^{\pm}$ as the orbit parameters *outside* the sphere $|q_-| = 2\mu^{\kappa}$ and add a subscript *in* to the Delaunay variables *inside* the sphere. We relate \mathcal{C}^0 estimates of Lemma 10.1 to the \mathcal{C}^1 estimates obtained above. Namely consider the following equation which is obtained by discarding the $o(1)$ errors in (10.1)

$$(10.28) \quad q_-^+ = 0, \quad p_-^+ = \text{Rot}(\alpha)p_-^-, \quad \mathbf{q}^+ = \mathbf{q}^-, \quad \mathbf{p}^+ = \mathbf{p}^-,$$

where α is given in (10.2). We have the following corollary saying that $d\mathbb{L}$ can be obtained by taking derivative directly in (10.28).

Corollary 10.1. *The derivative of the local map has the following form*

$$(10.29) \quad d\mathbb{L} = \frac{1}{\mu} (\hat{\mathbf{u}} + O(\mu^{\kappa})) \otimes \mathbf{1} + \hat{B} + O(\mu^{3\kappa-1}),$$

where $\hat{\mathbf{u}}, \mathbf{1}$ and \hat{B} are computed from (10.28). In particular,

$$\begin{aligned}
(10.30) \quad \hat{\mathbf{u}} &= \frac{\partial \mathcal{V}^+}{\partial \mathcal{X}^+} \frac{\partial \mathcal{X}^+}{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^+} \frac{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^+}{\partial \alpha} \left(\mu \frac{\partial \alpha}{\partial G_{in}} \right), \\
\mathbf{1} &= \frac{\partial G_{in}}{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^-} \frac{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^-}{\partial \mathcal{X}^-} \frac{\partial \mathcal{X}^-}{\partial \mathcal{V}^-}.
\end{aligned}$$

Proof. In (10.30), the derivatives $\frac{\partial \mathcal{V}^+}{\partial \mathcal{X}^+} \frac{\partial \mathcal{X}^+}{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^+}$ in $\hat{\mathbf{u}}$ and $\frac{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^-}{\partial \mathcal{X}^-} \frac{\partial \mathcal{X}^-}{\partial \mathcal{V}^-}$ in $\mathbf{1}$ are obvious. We focus on the remaining part.

We have

$$\frac{\partial(\mathbf{q}, \mathbf{p})^+}{\partial(\mathbf{q}, \mathbf{p})^-} = \text{Id}_4, \quad \frac{\partial(\mathbf{q}, \mathbf{p})^+}{\partial(q_-, p_-)^-} = \frac{\partial(q_-, p_-)^+}{\partial(\mathbf{q}, \mathbf{p})^-} = 0, \quad \frac{\partial(\mathbf{q}, \mathbf{p})^+}{\partial\alpha} = \frac{\partial G_{in}}{\partial(\mathbf{q}, \mathbf{p})^-} = 0$$

from (10.28) and (10.2) for G_{in} , which agrees with the corresponding blocks in (10.27) up to an $o(1)$ error as $\mu \rightarrow 0$. It remains to compare $\frac{\partial(q_-, p_-)^+}{\partial(q_-, p_-)^-}$.

It is easy to see from (10.24) that the expression for \mathbf{l} in (10.30) is true.

We take derivative directly in (10.28) to get $\frac{\partial(q_-, p_-)^+}{\partial\alpha} = \left(0, \frac{\partial p_-^+}{\partial\alpha}\right)$. To get the expression of $\hat{\mathbf{u}}$ in (10.30), it is enough to show the following compared with (10.24)

$$(10.31) \quad \frac{\partial p_-^+}{\partial\alpha} \left(\frac{\partial\alpha}{\partial G_{in}^-} \right) = \left(\frac{\partial p_-^+}{\partial G_{in}^+} + \frac{\partial \arctan \frac{G_{in}^-}{\mu \mathcal{L}_{in}^-}}{\partial G_{in}^-} \frac{\partial p_-^+}{\partial g_{in}^+} \right),$$

Actually we have using (10.3) and geometric consideration

$$p_-^+ = \text{Rot}(\alpha) p_-^- + O(e^{-2|u|}) = \text{Rot}(\beta) (|p_-^-|, 0) + O(e^{-2|u|}), \quad e^{-|u|} \simeq \mu^\kappa, \quad \text{where}$$

$$\alpha = \pi - 2 \arctan \frac{G_{in}}{\mu \mathcal{L}}, \quad \beta = g_{in} - \arctan \frac{G_{in}}{\mu \mathcal{L}_{in}}, \quad g = \pi - \arctan \frac{G_{in}}{\mu \mathcal{L}_{in}} + \arctan \frac{p_{-, \perp}^-}{p_{-, \parallel}^-} + O(\mu^{2\kappa}).$$

The angle g is formed by the x -axis with the symmetric axis of the hyperbola pointing to the opening. We take the G_{in} derivative directly and neglect $e^{-2|u|}$ term in the p_-^+ expression above to get (10.31). The $e^{-2|u|}$ term is negligible as we did in the proof of Lemma 3.1. In (10.28), p_-^+ also depends on p_- explicitly. When we take partial derivative with respect to the explicit dependence, we get a $O(1)$ matrix that goes into \hat{B} . We again compare with (10.24) to show the equivalence of \hat{B} obtained in two different ways. However, we will not need any information from \hat{B} except its boundedness in the paper. The proof is now complete. \square

Corollary 10.2. *Let $\gamma(s) : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^{10}$ be a \mathcal{C}^1 curve in the phase space such that $\Gamma = \gamma'(0) = O(1)$ and $\frac{\partial G_{in}^-}{\partial s} = O(\mu)$ then when taking derivative with respect to s in equations*

$$\begin{cases} |p_3^+|^2 + |p_4^+|^2 = |p_3^-|^2 + |p_4^-|^2 + o(1), \\ (\mathbf{q}, \mathbf{p})^+ = (\mathbf{q}, \mathbf{p})^- + o(1), \end{cases}$$

obtained from equation (10.1), the $o(1)$ terms are small in the \mathcal{C}^1 sense.

Proof. For the motion of the mass center, it follows from Corollary 10.1 that

$$\frac{\partial(\mathbf{q}, \mathbf{p})^+}{\partial(q_-, p_-, \mathbf{q}, \mathbf{p})^-} = \frac{1}{\mu} \frac{\partial(\mathbf{q}, \mathbf{p})^+}{\partial\alpha} \otimes \mathbf{l} + (0_{4 \times 4}, \text{Id}_{4 \times 4}) + o(1). \quad \text{We already obtained that}$$

$$\frac{\partial(\mathbf{q}, \mathbf{p})^+}{\partial\alpha} = O(\mu^{2\kappa}) \quad (\text{see equation (10.27)}). \quad \text{Due to Corollary 10.1 our assumption}$$

$$\text{that } \frac{\partial G_{in}^-}{\partial s} = O(\mu) \text{ implies that}$$

$$(10.32) \quad \mathbf{l} \cdot \Gamma = O(\mu)$$

which suppresses the $1/\mu$ term. This proves the corollary for the last two identities. To derive the first equation we use the fact that the Hamiltonian (4.9) is preserved. Namely we use the fact that RHS (4.9) is the same in $+$ and $-$ variables. It is enough to show $\frac{d}{ds}(|p_+^+|^2 - |p_-^+|^2) = o(1)$ since we already have the required estimate for the velocity of the mass center. In (4.9), the terms involving only \mathbf{q}, \mathbf{p} are handled using the result of the previous paragraph. The term $-\frac{\mu}{|q_-|}$ vanishes when taking derivative since $|q_-| = 2\mu^\kappa$ is constant. All the remaining terms have q_- to the power 2 or higher. We have $\frac{\partial q_-}{\partial s} = O(1)$ since $\Gamma = O(1)$. We also have $\frac{\partial q_+}{\partial s} = O(1)$ due to (10.32). Therefore after taking the s derivative, any term involving q_- is of order $O(\mu^\kappa)$. This completes the proof of the energy conservation part. \square

10.4. Proof of Lemma 2.2. Now we are ready to prove Lemma 2.2.

Proof of Lemma 2.2. We first convert the variables $(E_3, \ell_3, e_3, g_3; x_1, v_1; e_4, g_4)$ to $(x_3, v_3; x_1, v_1; x_4, v_4)^R$, and then to the variables $(p_3, q_3; p_1, q_1; p_4, q_4)$ using (2.3). Notice that (2.3) is a $O(\mu)$ perturbation of Id. The $O(\mu)$ difference is absorbed into $o(1)$ in the statement of the Lemma. It is enough to show that in the coordinates $(p_3, q_3; p_1, q_1; p_4, q_4)$, the (p_3, q_3, p_4, q_4) components of the local map is $o(1)$ close to Gerver's map \mathbf{G} as $1/\chi \ll \mu \rightarrow 0$. We know that \mathbf{G} is defined through elastic collision. Lemma 10.1 shows that the (p_3, q_3, p_4, q_4) components of the local map is also $o(1)$ close to elastic collision as $\mu \rightarrow 0$. Lemma 10.1 also shows that the outgoing information, i.e. the variables with superscript $+$ is determined by the incoming information ($-$ variables) and the rotation angle α up to an $o(1)$ error as $\mu \rightarrow 0$.

In the following proof, we show that the elastic collision process for \mathbf{G} and \mathbb{L} has the same incoming variables and same α up to an $o(1)$ error as $\mu \rightarrow 0$. For \mathbf{G} , the incoming orbit parameters are $(E_3, \ell_3, e_3, g_3, e_4, g_4)^-$, where we eliminate E_4^- since we have $E_3^- = -E_4^-$ in Gerver's case. The phase variables ℓ_3^-, ℓ_4^- are determined by the remaining shape variables $(E, e, g)_{3,4}^-$ since the intersection point of ellipse and hyperbola is determined by the shape variables. Now in the statement of the Lemma, we have that \mathbb{L} and \mathbf{G} have the same values for E_3^-, e_3^-, g_3^- as well as e_4^- . It remains to consider E_4^-, g_4^- and α . We first have that the initial value E_4^- of \mathbb{L} is $O(\mu)$ close to $-E_3^-$ following from (AL.3). Indeed (AL.3) implies that the energy of (x_1, v_1) is $O(\mu)$, so the total energy conservation apply to the zero energy level implies that $L_4^- = L_3^- + O(\mu)$ (see the Hamiltonian (4.3)).

It remains to consider g_4 and α . We replace g_4^-, g_4^+ by $\theta^-, \bar{\theta}^+$ using

$$(10.33) \quad \bar{\theta}^+ = g_4^+ - \text{sign}(u) \arctan \frac{G_4^+}{L_4^+}, \quad \theta^- = g_4^- - \text{sign}(u) \arctan \frac{G_4^-}{L_4^-}.$$

The assumption (AL.2) on θ^- implies that the initial value g_4^- for \mathbb{L} is $O(\mu)$ close to that of \mathbf{G} (horizontal asymptotes). We next express $\bar{\theta}^+$ using Cartesian coordinates $(p_4, q_4)^+$ through $(G_4, L_4, g_4)^+$. Next we apply Lemma 10.1 to $(p_4, q_4)^+$ to express $\bar{\theta}^+$ as a function of $\alpha, p_3^-, q_3^-, p_4^-, q_4^-$ up to an $o(1)$ error. Our initial value for \mathbb{L} differs from that of \mathbf{G} by $O(\mu)$ as we saw above. So we substitute those value of $p_3^-, q_3^-, p_4^-, q_4^-$ to get that $\bar{\theta}^+$ is a function of α only with an $o(1)$ error as $\mu \rightarrow 0$. Since the outgoing asymptote satisfies $|\bar{\theta}^+ - \pi| < \bar{\theta}$ as assumed, we get that α is

$o(1)$ close to Gerver's case as $\mu, \tilde{\theta} \rightarrow 0$ using implicit function theorem. To use implicit function theorem, we need to check $\frac{d\bar{\theta}^+}{d\alpha} \neq 0$. We know $d\bar{\theta}^+ = L_4^+ \bar{\mathbf{l}}$ from Remark 3.2 or from (10.33) directly and $\frac{\partial \mathcal{V}^+}{\partial \alpha} = c\hat{\mathbf{u}} + o(1)$ from Corollary 10.1 where $c^{-1} = \mu \frac{\partial \alpha}{\partial G_{in}^-}$ (see (10.2) and Corollary 10.1) is a constant that is bounded and independent of μ . We have $\frac{d\bar{\theta}^+}{d\alpha} = cL_4^+ \bar{\mathbf{l}} \cdot \hat{\mathbf{u}} + o(1) \neq 0$ by part (a) of Lemma 3.4. We use Lemma 10.1 again to get that $(p_3, q_3, p_4, q_4)^+$ is $o(1)$ close to Gerver's case as $\mu, \tilde{\theta} \rightarrow 0$. The proof is completed after converting Cartesian variables to Delaunay variables. \square

10.5. Proof of Lemma 3.4. In this section we work out the $O(1/\mu)$ term in the local map.

Proof. The proof relies on a numerical computation.

Before collision, $\hat{\mathbf{l}} = \frac{\partial G_{in}}{\partial \mathcal{V}^-}$. According to Corollary 10.1 we can differentiate the asymptotic expression of Lemma 10.1. We have $\left(\frac{\partial G_{in}}{\partial G_4^-}, \frac{\partial G_{in}}{\partial g_4^-} \right) =$

$$-(p_3^- - p_4^-) \times \left(\frac{\partial}{\partial G_4^-}, \frac{\partial}{\partial g_4^-} \right) q_4 - (p_3^- - p_4^-) \times \left(\frac{\partial q_4}{\partial \ell_4^-} \right) \cdot \left(\frac{\partial \ell_4^-}{\partial G_4^-}, \frac{\partial \ell_4^-}{\partial g_4^-} \right) + O(\mu^\kappa + \mu^{1-2\kappa}),$$

where $O(\mu^\kappa)$ comes from $\left(\frac{\partial}{\partial \mathcal{V}^-} (p_3^- - p_4^-) \right) \times (q_3 - q_4)$ and $O(\mu^{1-2\kappa})$ comes from $\frac{\partial q_4}{\partial L_4^-} \frac{\partial L_4^-}{\partial -}$ where L_4 is solved from the Hamiltonian (4.7) $H = 0$.

We need to eliminate ℓ_4 using the relation $|q_3 - q_4| = \mu^\kappa$.

$$\begin{aligned} \left(\frac{\partial \ell_4^-}{\partial G_4^-}, \frac{\partial \ell_4^-}{\partial g_4^-} \right) &= - \left(\frac{\partial |q_3 - q_4|}{\partial \ell_4^-} \right)^{-1} \left(\frac{\partial |q_3 - q_4|}{\partial G_4^-}, \frac{\partial |q_3 - q_4|}{\partial g_4^-} \right) \\ &= - \frac{(q_3 - q_4) \cdot \left(\frac{\partial q_4}{\partial G_4^-}, \frac{\partial q_4}{\partial g_4^-} \right)}{(q_3 - q_4) \cdot \frac{\partial q_4}{\partial \ell_4^-}} = - \frac{(p_3^- - p_4^-) \cdot \left(\frac{\partial q_4}{\partial G_4^-}, \frac{\partial q_4}{\partial g_4^-} \right)}{(p_3^- - p_4^-) \cdot \frac{\partial q_4}{\partial \ell_4^-}} + O(\mu^{1-\kappa}). \end{aligned}$$

Here we replaced $q_3^- - q_4^-$ by $p_3^- - p_4^-$ using the fact that the two vectors form an angle of order $O(\mu^{1-\kappa})$ (see Lemma 10.1(c)). Therefore

$$\begin{aligned} \left(\frac{\partial G_{in}}{\partial G_4^-}, \frac{\partial G_{in}}{\partial g_4^-} \right) &= -(p_3^- - p_4^-) \times \left(\frac{\partial}{\partial G_4^-}, \frac{\partial}{\partial g_4^-} \right) q_4 \\ &\quad + (p_3^- - p_4^-) \times \frac{\partial q_4}{\partial \ell_4^-} \left(\frac{(p_3^- - p_4^-) \cdot \left(\frac{\partial q_4}{\partial G_4^-}, \frac{\partial q_4}{\partial g_4^-} \right)}{(p_3^- - p_4^-) \cdot \frac{\partial q_4}{\partial \ell_4^-}} \right) + O(\mu^\kappa + \mu^{1-2\kappa}). \end{aligned}$$

Similarly, we get

$$\frac{\partial G_{in}}{\partial \ell_3^-} = (p_3^- - p_4^-) \times \frac{\partial q_3}{\partial \ell_3^-} + \left((p_3^- - p_4^-) \times \frac{\partial q_4}{\partial \ell_4^-} \right) \left(\frac{(p_3^- - p_4^-) \cdot \frac{\partial q_3}{\partial \ell_3^-}}{(p_3^- - p_4^-) \cdot \frac{\partial q_4}{\partial \ell_4^-}} \right) + O(\mu^\kappa + \mu^{1-2\kappa}).$$

We use MATHEMATICA and the data in the Appendix B.2 to work out $\frac{\partial G_{in}}{\partial \mathcal{V}^-}$. The results are : for the first collision, $\hat{\mathbf{l}}_1 = [*, -0.8, *, *, 0, 0, 0, 0, 3.42, -2.54]$, and for the second collision: $\hat{\mathbf{l}}_2 = [*, -0.35, *, *, 0, 0, 0, 0, 3.44, -0.47]$. Entries with $*$ are not needed. We can check directly that $\hat{\mathbf{l}}_i \cdot w_{3-i} \neq 0$ and $\hat{\mathbf{l}}_i \cdot \tilde{w} \neq 0$ for $i = 1, 2$ using the expressions of w, \tilde{w} in Lemma 3.2.

After collision, $\hat{\mathbf{u}} = \frac{\partial \mathcal{V}^+}{\partial \alpha}$. In equation (10.1), we let $\mu \rightarrow 0$. Applying the implicit function theorem to (10.1) with $\mu = 0$ we obtain

$$\begin{aligned} & \left(\frac{\partial \mathcal{X}^+}{\partial \mathcal{V}^+} + \frac{\partial \mathcal{X}^+}{\partial \ell_4^+} \otimes \frac{\partial \ell_4^+}{\partial \mathcal{V}^+} \right) \cdot \frac{\partial \mathcal{V}^+}{\partial \alpha} \\ &= \frac{1}{2} \left(0, 0, \text{Rot} \left(\frac{\pi}{2} + \alpha \right) (p_3^- - p_4^-); 0, 0, 0, 0; 0, 0, -\text{Rot} \left(\frac{\pi}{2} + \alpha \right) (p_3^- - p_4^-) \right)^T \\ &= \frac{1}{2} \left(0, 0, \text{Rot} \left(\frac{\pi}{2} \right) (p_3^+ - p_4^+); 0, 0, 0, 0; 0, 0, -\text{Rot} \left(\frac{\pi}{2} \right) (p_3^+ - p_4^+) \right)^T. \end{aligned}$$

where $\text{Rot}(\pi/2 + \alpha) = \frac{d\text{Rot}(\alpha)}{d\alpha}$ and $\frac{\partial \ell_4^+}{\partial \mathcal{V}^+}$ is given by (10.9). Again we use MATHEMATICA to work out the $\frac{\partial \mathcal{V}^+}{\partial \alpha}$. The results are:

for the first collision $\hat{\mathbf{u}}_1 = [-0.49, *, *, *, 0, 0, 0, 0, -0.20, -0.64]$ and for the second collision $\hat{\mathbf{u}}_2 = [-1.00, *, *, *, 0, 0, 0, 0, 0.34, -0.50]$. We can check directly that $\bar{\mathbf{l}}_j \cdot \hat{\mathbf{u}}_j \neq 0$ for $j = 1, 2$ using $\bar{\mathbf{l}}$ in Lemma 3.2.

To obtain a symbolic sequence with any order of symbols 3, 4 as claimed in the main theorem, we notice that the only difference is that the outgoing relative velocity changes sign $(p_3^+ - p_4^+) \rightarrow -(p_3^+ - p_4^+)$. So we only need to send $\hat{\mathbf{u}} \rightarrow -\hat{\mathbf{u}}$. \square

10.6. Proof of Lemma 3.5. In this section, we prove Lemma 3.5, which guarantees the non degeneracy condition Lemma 3.3 (see the proof of Lemma 3.3). Since we have already obtained \mathbf{l} and \mathbf{u} in $d\mathbb{L}$ and $\bar{\mathbf{l}}, \bar{\mathbf{l}}, \bar{\mathbf{u}}, \bar{\mathbf{u}}$ in $d\mathbb{G}$, one way to prove Lemma 3.3 is to work out the matrix B explicitly using Corollary 10.1 on computer. In that case, the current section is not necessary. However, in this section, we use a different approach, which simplifies the computation and has several advantages. The first advantage is that this treatment has clear physical and geometrical meaning. Second, we use the same way to control the shape of the ellipse in Appendix B.3. Third, this method gives us a way to deal with the singular limit $d\mathbb{L}$ as $\mu \rightarrow 0$. Recall that Lemmas 3.1 and 3.2 give the following form for the derivatives of local map and global maps

$$d\mathbb{L} = \frac{1}{\mu} \mathbf{u}_j \otimes \mathbf{l}_j + B + O(\mu^\kappa), \quad d\mathbb{G} = \chi^2 \bar{\mathbf{u}}_j \otimes \bar{\mathbf{l}}_j + \chi \bar{\mathbf{u}}_j \otimes \bar{\bar{\mathbf{l}}}_j + O(\mu^2 \chi),$$

where $j = 1, 2$ standing for the first or second collision. Moreover, in the limit $\chi \rightarrow \infty, \mu \rightarrow 0$,

$$\text{span}\{\bar{\mathbf{u}}_j, \bar{\bar{\mathbf{u}}}_j\} \rightarrow \text{span}\{w_j, \tilde{w}\}, \quad \mathbf{l}_j \rightarrow \hat{\mathbf{l}}_j, \quad \bar{\mathbf{l}}_j \rightarrow \hat{\bar{\mathbf{l}}}_j, \quad \bar{\bar{\mathbf{l}}}_j \rightarrow \hat{\hat{\bar{\mathbf{l}}}}_j, \quad j = 1, 2.$$

We first prove an abstract lemma that reduces the study of the local map of the $\mu > 0$ case to $\mu = 0$ case. It shows that we can find a direction in $\text{span}\{\bar{\mathbf{u}}, \bar{\bar{\mathbf{u}}}\}$, along which the directional derivative of $d\mathbb{L}$ is not singular.

Lemma 10.3. *Suppose the vector $\tilde{\Gamma}_\mu \in \text{span}\{\bar{\mathbf{u}}_{3-j}, \bar{\mathbf{u}}_{3-j}\}$ satisfies $\bar{\mathbf{l}}_j(d\mathbb{L}\tilde{\Gamma}_\mu) = 0$ and $\|\tilde{\Gamma}_\mu\|_\infty = 1$. Then we have $\mathbf{l}_j(\tilde{\Gamma}_\mu) = O(\mu)$ as $\mu \rightarrow 0$ and the following limits exist*

$$\Gamma_{3-j} = \lim_{\mu \rightarrow 0} \tilde{\Gamma}_\mu \text{ and } \lim_{\mu \rightarrow 0} d\mathbb{L}\tilde{\Gamma}_\mu = \Delta_j,$$

and the Δ_j satisfies $\hat{\mathbf{l}}_j(\Delta_j) = 0$.

Proof. Denote $\Gamma'_\mu = \mathbf{l}_j(\bar{\mathbf{u}}_{3-j})\bar{\mathbf{u}}_{3-j} - \mathbf{l}_j(\bar{\mathbf{u}}_{3-j})\bar{\mathbf{u}}_{3-j} \in \text{Ker}\mathbf{l}_j$ and let v_μ be a vector in $\text{span}(\bar{\mathbf{u}}_{3-j}, \bar{\mathbf{u}}_{3-j})$ such that $v_\mu \rightarrow v$ as $\mu \rightarrow 0$ and $\mathbf{l}_j(v_\mu) = 1$. Suppose that

$$\tilde{\Gamma}_\mu = a_\mu v_\mu + b_\mu \Gamma'_\mu$$

then

$$(10.34) \quad d\mathbb{L}(\tilde{\Gamma}_\mu) = \frac{a_\mu}{\mu} \mathbf{l}_j(v_\mu) \mathbf{u}_j + a_\mu B_j(v_\mu) + b_\mu B_j \Gamma'_\mu + o(1).$$

So $\bar{\mathbf{l}}_j(d\mathbb{L}(\tilde{\Gamma}_\mu)) = 0$ implies that

$$(10.35) \quad a_\mu = -\mu \frac{b_\mu \bar{\mathbf{l}}_j(B_j \Gamma'_\mu) + o(1)}{\mathbf{l}_j(v_\mu) \bar{\mathbf{l}}_j(\mathbf{u}_j) + \mu \bar{\mathbf{l}}_j B_j(v_\mu)}.$$

The denominator is not zero since $\mathbf{l}_j(v_\mu) = 1$ and $\bar{\mathbf{l}}_j(\mathbf{u}_j)$ using Lemma 3.4. Therefore $a_\mu = O(\mu)$ and hence $\tilde{\Gamma}_\mu = b_\mu \Gamma'_\mu + O(\mu)$ and $\mathbf{l}_j(\tilde{\Gamma}_\mu) = O(\mu)$. Now the remaining statements of the lemma follow from equations (10.34) and (10.35). \square

To compute the numerical values it is more convenient for us to work with polar coordinates. We need the following quantities.

Definition 10.2. • ψ : polar angle, related to u by $\tan \frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}$ for ellipse. We choose the positive y axis as the axis $\psi = 0$. E : energy; e : eccentricity; G : angular momentum, g : argument of periapsis.
• The subscripts 3, 4 stand for q_3 or q_4 . The superscript \pm refers to before or after collision. Recall that all quantities are evaluated on the sphere

$$|q_3 - q_4| = \mu^\kappa.$$

Recall the formula $r = \frac{G^2}{1 - e \cos \psi}$ for conic sections in which the perigee lies on the axis $\psi = \pi$. In our case we have

$$(10.36) \quad \begin{cases} r_3^\pm = \frac{(G_3^\pm)^2}{1 - e_3^\pm \sin(\psi_3^\pm + g_3^\pm)} + o(1), \\ r_4^\pm = \frac{(G_4^\pm)^2}{1 - e_4^\pm \sin(\psi_4^\pm - g_4^\pm)} + o(1). \end{cases}$$

$o(1)$ terms are small when $1/\chi \ll \mu \rightarrow 0$.

Lemma 10.4. *Under the assumptions of Corollary 10.2 we have*

$$\frac{dr_3^+}{ds} = \frac{dr_4^+}{ds} + o(1), \quad \frac{dr_3^-}{ds} = \frac{dr_4^-}{ds} + o(1), \quad \frac{d\psi_3^+}{ds} = \frac{d\psi_4^+}{ds} + o(1), \quad \frac{d\psi_3^-}{ds} = \frac{d\psi_4^-}{ds} + o(1).$$

Moreover in (10.36) the $o(1)$ terms are also \mathcal{C}^1 small when taking the s derivative.

Proof. To prove the statement about (10.36), we use the Hamiltonian (4.7). The $r_{3,4}$ obey the Hamiltonian system (4.7). The estimate (10.8) shows the $\frac{-\mu}{|q_3 - q_4|}$ gives small perturbation to the variational equations. The two $O(1/\chi)$ terms in (4.7) are also small. This shows that the perturbations to Kepler motion is \mathcal{O}^1 small.

Next we consider the derivatives $\frac{\partial r_{3,4}^\pm}{\partial s}$. We consider first the case of “-”. From the condition $|\vec{r}_3 - \vec{r}_4| = \mu^\kappa$, for the Poincaré section we get

$$(\vec{r}_3 - \vec{r}_4) \cdot \frac{d}{ds}(\vec{r}_3 - \vec{r}_4) = 0.$$

This implies $(\vec{r}_3 - \vec{r}_4) \perp \frac{d}{ds}(\vec{r}_3 - \vec{r}_4)$.

We also know the angular momentum for the relative motion is

$$G_{in} = (\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \times (\vec{r}_3 - \vec{r}_4) = O(\mu),$$

which implies $\dot{\vec{r}}_3 - \dot{\vec{r}}_4$ is almost parallel to $\vec{r}_3 - \vec{r}_4$. The condition $\frac{\partial G_{in}^-}{\partial s} = O(\mu)$ reads

$$\left(\frac{d}{ds}(\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \right) \times (\vec{r}_3 - \vec{r}_4) + (\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \times \left(\frac{d}{ds}(\vec{r}_3 - \vec{r}_4) \right) = O(\mu).$$

Since the first term is $O(\mu^\kappa)$ due to our choice of the Poincaré section we see that

$$(\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \times \left(\frac{d}{ds}(\vec{r}_3 - \vec{r}_4) \right) = o(1).$$

Since $\frac{d}{ds}(\vec{r}_3 - \vec{r}_4)$ is almost perpendicular to $(\dot{\vec{r}}_3 - \dot{\vec{r}}_4)$ by the analysis presented above we get $\frac{d}{ds}(\vec{r}_3 - \vec{r}_4) = o(1)$. Taking the radial and angular part of this vector identity and using that $r_4 = r_3 + o(1)$, $\psi_4 = \psi_3 + o(1)$ we get “-” part of the lemma.

To repeat the above argument for “+” variables, we first need to establish $\frac{\partial G_{in}^-}{\partial s} = O(\mu)$. Indeed, using equations (10.15) and (10.25) we get

$$\begin{aligned} \frac{\partial G_{in}^+}{\partial \psi} &= \frac{\partial G_{in}^+}{\partial(\mathcal{L}_{in}T, G_{in}, g_{in}, \mathbf{q}, \mathbf{p})^-} \frac{\partial(\mathcal{L}_{in}, G_{in}, g_{in}, \mathbf{q}, \mathbf{p})^-}{\partial \psi} \\ &= O(\mu^{3\kappa}, 1, \mu^{3\kappa}, \mu_{1 \times 2}^{3\kappa}, \mu_{1 \times 2}^{3\kappa}) \cdot O(1, \mu, 1, 1_{1 \times 2}, 1_{1 \times 2}) = O(\mu). \end{aligned}$$

It remains to show $\left(\frac{d}{ds}(\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \right) = O(1)$ in the “+” case. Since we know it is true in the “-” case, the “+” case follows, because the directional derivative of the local map $d\mathbb{L}\Gamma$ is bounded due to our choice of Γ . \square

We are now ready to describe the computation of Lemma 3.5. The reader may notice that the computations in the proofs of Lemmas 3.5 and 2.1 are quite similar. Note however that Lemma 3.5 describes the *subleading* term for the derivative of the local map. By contrast the *leading* term can not be understood in terms of the Gerver map since it comes from the possibility of varying the closest distance between q_3 and q_4 and this distance is assumed to be zero in Gerver’s model.

We will use the following set of equations which follows from (10.28).

$$(10.37) \quad E_3^+ + E_4^+ = E_3^- + E_4^-,$$

$$(10.38) \quad G_3^+ + G_4^+ = G_3^- + G_4^-,$$

$$(10.39) \quad \frac{e_3^+}{G_3^+} \cos(\psi_3^+ + g_3^+) + \frac{e_4^+}{G_4^+} \cos(\psi_4^+ - g_4^+) = \frac{e_3^-}{G_3^-} \cos(\psi_3^- + g_3^-) + \frac{e_4^-}{G_4^-} \cos(\psi_4^- - g_4^-),$$

$$(10.40) \quad \frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi_3^+ + g_3^+)} = \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)},$$

$$(10.41) \quad \psi_3^+ = \psi_3^-,$$

$$(10.42) \quad \frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi_3^+ + g_3^+)} = \frac{(G_4^+)^2}{1 - e_4^+ \sin(\psi_4^+ - g_4^+)},$$

$$(10.43) \quad \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi_4^- - g_4^-)},$$

$$(10.44) \quad \psi_4^- = \psi_3^-,$$

$$(10.45) \quad \psi_4^+ = \psi_3^+,$$

In the above equations we have dropped $o(1)$ terms for brevity. We would like to emphasize that the above approximations hold not only in \mathcal{C}^0 sense but also in \mathcal{C}^1 sense when we take the derivatives along the directions satisfying the conditions of Corollary 10.2. (10.37) is the conservation of the energy, (10.38) is the conservation of the angular momentum and (10.39) follows from the momentum conservation (see the derivation of (B.2) in Appendix B.3). The possibility of differentiating these equations is justified in Corollary 10.2. The remaining equations reflect the fact that q_3^\pm and q_4^\pm are all close to each other. The possibility of differentiating these equations is justified by Lemma 10.4.

We set the total energy to be zero. So we get $E_4^\pm = -E_3^\pm$. This eliminates E_4^\pm . Then we also eliminate ψ_4^\pm by setting them to be equal ψ_3^\pm .

Proof of the Lemma 3.5. Lemma 10.3 and Corollary 10.1 show that the assumption of Lemma 3.5 implies that the direction Γ along which we take the directional derivative satisfies $\frac{\partial G_{in}}{\partial \Gamma} = O(\mu)$. So we can directly take derivatives in equations (10.37)-(10.37). Recall that we need to compute $dE_3^+(d\mathbb{L}\Gamma)$ where $\Gamma \in \text{Ker} \mathbf{l}_j \cap \text{span}\{w_{3-j}, \tilde{w}\}$. Lemma 3.2 tells us that in Delaunay coordinates we have

$$(10.46) \quad \tilde{w} = (0, 1, 0_{1 \times 8}), \quad w = (0_{1 \times 8}, 1, a) \text{ where } a = \frac{-L_4^-}{(L_4^-)^2 + (G_4^-)^2}.$$

The formula $\tan \frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}$ which relates ψ to ℓ through u shows that (10.46) also holds if we use $(L_3, \psi_3, G_3, g_3; x_1, v_1; G_4, g_4)$ as coordinates. Hence Γ has the form $(0, 1, 0_{1 \times 6}, c, ca)$. To find the constant c we use (10.43). Since the (x_1, v_1) components in Γ are zero and we see from (10.27) that the (x_1, v_1)

components of $d\mathbb{L}\Gamma$ is $O(\mu^\kappa) \rightarrow 0$ using $\frac{\partial G_{in}}{\partial \Gamma} = O(\mu)$. We can eliminate $(x_1, v_1)^\pm$ from our list of variables.

Note that the expression $dE_3^+(d\mathbb{L}\Gamma)$ does not involve $d\psi_3^+$. Therefore we can eliminate ψ_3^+ from consideration by setting $\psi_3^+ = \psi_3^- = \psi$ (see (10.41)). Let \mathbf{L} denote the projection of our map to $(L_3, G_3, g_3, G_4, g_4)$ variables. Thus we need to find $dE_3^+(d\mathbf{L}\Gamma)$. To this end write the remaining equations ((10.38), (10.39), (10.40), and (10.42)) formally as $\mathbf{F}(Z^+, Z^-) = 0$, where in $Z^+ = (E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)$ and $Z^- = (E_3^-, \psi, G_3^-, g_3^-, G_4^-, g_4^-)$.

We have

$$\frac{\partial \mathbf{F}}{\partial Z^+} d\mathbf{L}\Gamma + \frac{\partial \mathbf{F}}{\partial Z^-} \Gamma = 0.$$

However, $\frac{\partial \mathbf{F}}{\partial Z^+}$ is not invertible since \mathbf{F} involves only four equations of \mathbf{F} while Z^+ has 5 variables. To resolve this problem we use that by definition of Γ we have $\bar{\mathbf{I}} \cdot \frac{\partial Z^+}{\partial \psi} = 0$, where $\bar{\mathbf{I}} = \left(\frac{G_4^+/L_4^+}{(L_4^+)^2 + (G_4^+)^2}, 0_{1 \times 3}, \frac{-1}{(L_4^+)^2 + (G_4^+)^2}, \frac{1}{L_4^+} \right)$ by Lemma 3.2. Thus we get

$$\begin{bmatrix} \bar{\mathbf{I}} \\ \frac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix} d\mathbf{L}\Gamma = - \begin{bmatrix} 0 \\ \frac{\partial \mathbf{F}}{\partial Z^-} \Gamma \end{bmatrix} \implies d\mathbf{L}\Gamma = - \begin{bmatrix} \bar{\mathbf{I}} \\ \frac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{\partial \mathbf{F}}{\partial Z^-} \Gamma \end{bmatrix}.$$

We use computer to complete the computation. We only need the first entry $\frac{\partial E_3^+}{\partial \psi}$ to prove Lemma 3.5. It turns out this number is 1.855 for the first collision and -1.608 for the second collision. Both are nonzero as needed. \square

11. REMARKS ON NON-COLLISION SINGULARITIES IN N -BODY PROBLEM WITH $N \geq 5$

In this section, we briefly talk about the non-collision singularities in N -body problem with $N \geq 5$. The discussion here is only preliminary. To work out the argument here rigorously, one needs to work with larger matrices.

- (1) *The configuration.* The positions of Q_1, Q_2, Q_3, Q_4 are the same as this paper. We put extra bodies far from the system and moving away with an initial velocity almost perpendicular to the $Q_1 Q_2$ line.
- (2) *The coordinates.* We define $q_i = Q_i - Q_2$, $p_i = \frac{1}{\mu} P_i$, $i \neq 2$. Then we perform linear symplectic transformations $(p_i, q_i) \rightarrow (v_i, x_i)$ similar to (2.3) and (4.2) to reduce the Hamiltonian to a form similar to (4.3) and (4.5). This can always be done. The idea is to kill the inner product terms in the Hamiltonian (4.1). There are $\binom{N-1}{2} = (N-1)(N-2)/2$ inner product terms. The transformation for p_i part in (2.3) can be written as a upper triangular matrix having $(N-1)(N-2)/2$ undetermined entries. The left case (4.2) can also be determined similarly. We define the Poincaré sections in the same way as Definition 2.3. The resulting Hamiltonian has the form.

$$H = \left(\frac{v_1^2}{2m_1} - \frac{k_1}{|x_1|} \right) - \frac{m_3 k_3^2}{2L_3^2} + \frac{m_4 k_4^2}{2L_4^2} + \sum_{i=5}^N \left(\frac{v_i^2}{2m_i} - \frac{k_i}{|x_i|} \right) + \text{perturbation}.$$

We have the estimates $m_1, k_1, k_i \sim 1/\mu$, $i \geq 5$, $m_3, m_4, k_3, k_4 \sim 1$ and m_i , $i \geq 5$ can be chosen arbitrarily. If we take Taylor expansion of the perturbation at $x_1 = \infty$ using (4.8), we expect to get new monomials of the form $\frac{\langle x_1, x_i \rangle^m |x_i|^{2n}}{|x_1|^{2m+2n+1}}$, $m + 2n \geq 2$, $i \geq 5$ as leading terms using the same argument as Lemma 6.3.

- (3) *Rescaling.* We expect to have $|x_1| \sim \lambda^n(1 + c\mu)^n \chi_0$, $|x_i| \sim \lambda^n |x_i(0)|$ in the renormalized system. It takes time $\lambda^n(1 + c\mu)^n \chi_0$ for Q_4 to complete one return in the renormalized system and $\lambda^{-n/2}(1 + c\mu)^n \chi_0$ in the unrenormalized system. This gives finite time blow up solutions.
- (4) *Total angular momentum conservation.* The total angular momentum is $\sum_{i \neq 2} G_i = \sum_{i \neq 2} v_i \times x_i$. We have that $G_3, G_4 \sim 1$ in the renormalized system and $\sim \lambda^{-n/2}$ in the unrenormalized system.

$$\dot{G}_i = \frac{\partial}{\partial g} \text{perturbation} \sim \frac{|x_i|}{|x_1|^2} \sim \lambda^{-n}(1 + c\mu)^{-2n} \chi_0^{-2}, \quad i \geq 5$$

and the total oscillation is $(1 + c\mu)^{-n} \chi_0^{-1}$ within one return of Q_4 in the renormalized system. So we expect G_i , $i \geq 5$ to approach a constant in the unrenormalized system and grows like λ^n in the renormalized system. The total angular momentum conservation therefore implies that G_1 behaves in the same way as in $N = 4$ case. So we use our result of this paper with nonzero total angular momentum.

- (5) *The local map.* The local map is the same as that in this paper as well as that in [DX].
- (6) *The global map.* The transformations (II), (IV) do not make much differences from the four-body case, neither the boundary contributions to (I), (III), (V). To estimate the variational equations and its solutions, we consider first 14×14 matrix of the five-body case. When we add more bodies, we treat them together with (x_5, v_5) as a large vector and the resulting matrix would be the same as the five-body case. We expect that Lemma 3.2 can be established since the hyperbolicity comes from mainly from Q_3 and Q_4 .

APPENDIX A. DELAUNAY COORDINATES

A.1. Elliptic motion. The material of this section could be found in [Al]. Consider the two-body problem with Hamiltonian

$$H(P, Q) = \frac{|P|^2}{2m} - \frac{k}{|Q|}, \quad (P, Q) \in \mathbb{R}^4.$$

This system is integrable in the Liouville-Arnold sense when $H < 0$. So we can introduce the action-angle variables (L, ℓ, G, g) in which the Hamiltonian can be written as

$$H(L, \ell, G, g) = -\frac{mk^2}{2L^2}, \quad (L, \ell, G, g) \in T^*\mathbb{T}^2.$$

The Hamiltonian equations are

$$\dot{L} = \dot{G} = \dot{g} = 0, \quad \dot{\ell} = \frac{mk^2}{L^3}.$$

We introduce the following notation

E -energy, M -angular momentum, e -eccentricity, a -semimajor, b -semiminor.

Then we have the following relations which explain the physical and geometrical meaning of the Delaunay coordinates.

$$a = \frac{L^2}{mk}, \quad b = \frac{LG}{mk}, \quad E = -\frac{k}{2a}, \quad M = G, \quad e = \sqrt{1 - \left(\frac{G}{L}\right)^2}.$$

Moreover, g is the argument of periapsis and ℓ is called the mean anomaly, and ℓ can be related to the polar angle ψ through the equations

$$\tan \frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{u}{2}, \quad u - e \sin u = \ell.$$

We also have the Kepler's law $\frac{a^3}{T^2} = \frac{1}{(2\pi)^2}$ which relates the semimajor axis a and the period T of the ellipse.

Denoting particle's position by (q_1, q_2) and its momentum (p_1, p_2) we have the following formulas in case $g = 0$.

$$\begin{cases} q_1 = a(\cos u - e), \\ q_2 = a\sqrt{1-e^2} \sin u, \end{cases} \quad \begin{cases} p_1 = -\sqrt{mka}^{-1/2} \frac{\sin u}{1-e \cos u}, \\ p_2 = \sqrt{mka}^{-1/2} \frac{\sqrt{1-e^2} \cos u}{1-e \cos u}, \end{cases}$$

where u and l are related by $u - e \sin u = \ell$.

Expressing e and a in terms of Delaunay coordinates we obtain the following

$$(A.1) \quad \begin{aligned} q_1 &= \frac{L^2}{mk} \left(\cos u - \sqrt{1 - \frac{G^2}{L^2}} \right), \quad q_2 = \frac{LG}{mk} \sin u. \\ p_1 &= -\frac{mk}{L} \frac{\sin u}{1 - \sqrt{1 - \frac{G^2}{L^2}} \cos u}, \quad p_2 = \frac{mk}{L^2} \frac{G \cos u}{1 - \sqrt{1 - \frac{G^2}{L^2}} \cos u}. \end{aligned}$$

Here g does not enter because the argument of perihelion is chosen to be zero. In general case, we need to rotate the (q_1, q_2) and (p_1, p_2) using the matrix $\begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix}$.

Notice that the equation (A.1) describes an ellipse with one focus at the origin and the other focus on the negative x -axis. We want to be consistent with [G2], i.e. we want $g = \pi/2$ to correspond to the ‘‘vertical’’ ellipse with one focus at the origin and the other focus on the positive y -axis (see Appendix B.2). Therefore we rotate the picture clockwise. So we use the Delaunay coordinates which are related to the Cartesian ones through the equation

$$(A.2) \quad \begin{aligned} Q_{\parallel} &= \frac{1}{mk} \left(L^2 \left(\cos u - \sqrt{1 - \frac{G^2}{L^2}} \right) \cos g + LG \sin u \sin g \right), \\ Q_{\perp} &= \frac{1}{mk} \left(-L^2 \left(\cos u - \sqrt{1 - \frac{G^2}{L^2}} \right) \sin g + LG \sin u \cos g \right). \end{aligned}$$

This is an ellipse focused at the origin with this other focus lying on the positive y axis.

A.2. Hyperbolic motion. The above formulas can also be used to describe hyperbolic motion, where we need to replace “ $\sin \rightarrow \sinh$, $\cos \rightarrow \cosh$ ” and change signs properly (c.f.[Al, F, W]). Namely, we have

$$(A.3) \quad \begin{aligned} a &= \frac{L^2}{mk}, \quad b = \frac{LG}{mk}, \quad E = \frac{k}{2L^2}, \quad M = G, \quad e = \sqrt{1 + \left(\frac{G}{L}\right)^2} \\ q_1 &= \frac{L^2}{mk} \left(\cosh u - \sqrt{1 + \frac{G^2}{L^2}} \right), \quad q_2 = \frac{LG}{mk} \sinh u, \\ p_1 &= -\frac{mk}{L} \frac{\sinh u}{1 - \sqrt{1 + \frac{G^2}{L^2}} \cosh u}, \quad p_2 = -\frac{mk}{L^2} \frac{G \cosh u}{1 - \sqrt{1 + \frac{G^2}{L^2}} \cosh u}. \end{aligned}$$

where u and ℓ are related by

$$(A.4) \quad u - e \sinh u = \ell, \quad \text{where } e = \sqrt{1 + \left(\frac{G}{L}\right)^2}.$$

This hyperbola is symmetric with respect to the x -axis, opens to the right and the particle moves clockwise on it when u increases (ℓ decreases). When the particle moves to the right of $x = -\frac{\chi}{2}$ line we have a hyperbola opening to the left and the particle moves anti-clockwise. To achieve this we first reflect (q_1, q_2) around the y -axis, then rotate it by an angle g , which is the angle formed by the x -axis with the symmetric axis of the hyperbola pointing to the opening. If we restrict $|g| < \pi/2$, then the particle moves anti-clockwise on the hyperbola as u increases (ℓ decreases) due to the reflection. Thus we have

$$(A.5) \quad \begin{aligned} Q_{\parallel} &= -\frac{1}{mk} (\cos g L^2 (\cosh u - e) + \sin g LG \sinh u), \\ Q_{\perp} &= \frac{1}{mk} (-\sin g L^2 (\cosh u - e) + \cos g LG \sinh u), \\ P &= \frac{mk}{1 - e \cosh u} \left(\frac{1}{L} \sinh u \cos g + \frac{G}{L^2} \sin g \cosh u, \right. \\ &\quad \left. \frac{1}{L} \sinh u \sin g - \frac{G}{L^2} \cos g \cosh u \right). \end{aligned}$$

We see from (A.4), when $|u|$ is large, we have $\text{sign}(u) = -\text{sign}(\ell)$. We have three difference choices of g in this paper.

- (a) If the incoming asymptote is horizontal, (see the arrows in Figure 1 and 2 for “incoming” and “outgoing”), then the particle comes from the left, and as u tends to $-\infty$, the y -coordinate is bounded and x -coordinate is negative. In this case we have $\tan g = -\frac{G}{L}$, $g \in (-\pi/2, 0)$.
- (b) If the outgoing asymptote is horizontal, then the particle escapes to the left, and as u tends to $+\infty$, the y -coordinate is bounded and x -coordinate is negative. In this case we have $\tan g = +\frac{G}{L}$, $g \in (0, \pi/2)$.

- (c) When the particle Q_4 is moving to the left of the sections $\{x_{4,\parallel}^R = -\chi/2\}$ and $\{x_{4,\parallel}^L = \chi/2\}$, we treat the motion as hyperbolic motion focused at Q_1 . We move the origin to Q_1 . The hyperbola opens to the right. The orbit has the following parametrization

$$(A.6) \quad \begin{aligned} Q &= \frac{1}{mk} (\cos gL^2(\cosh u - e) - \sin gLG \sinh u, \\ &\quad \sin gL^2(\cosh u - e) + \cos gLG \sinh u) . \\ P &= \frac{mk}{1 - e \cosh u} \left(-\frac{1}{L} \sinh u \cos g - \frac{G}{L^2} \sin g \cosh u, \right. \\ &\quad \left. -\frac{1}{L} \sinh u \sin g + \frac{G}{L^2} \cos g \cosh u \right) . \end{aligned}$$

We cite Lemma A.1 of [DX] to simplify our calculation. The lemma implies that we can replace $\pm u$ by $\ln(\mp \ell/e)$ when taking first and second order derivatives.

Lemma A.1 (Lemma A.1 of [DX]). *Let u be the function of ℓ, G and L given by (A.4). Then we can approximate u by $\ln(\mp \ell/e)$ in the following sense.*

$$u \mp \ln \frac{\mp \ell}{e} = O(\ln |\ell|/\ell), \quad \frac{\partial u}{\partial \ell} = \pm 1/\ell + O(1/\ell^2),$$

$$\left(\frac{\partial}{\partial L}, \frac{\partial}{\partial G} \right) (u \pm \ln e) = O(1/|\ell|), \quad \left(\frac{\partial}{\partial L}, \frac{\partial}{\partial G} \right)^2 (u \pm \ln e) = O(1/|\ell|),$$

Here the first sign is taken if $u > 0$ and the second sign is taken then $u < 0$. The estimates above are uniform as long as $|G| \leq K$, $1/K \leq L \leq K$, $\ell > \ell_0$ and the implied constants in $O(\cdot)$ depend only on K and ℓ_0 .

A.3. The derivative of Cartesian variables with respect to the Delaunay variables. Next, we calculate the first order derivatives of the Cartesian variables with respect to the Delaunay variables. The assumption of the next lemma is met by Lemma 6.9.

Lemma A.2. *Assume that $|G| \leq K$, $1/K \leq L \leq K$.*

- (a) *Assume further in the right case $g = \pm \arctan \frac{G}{L} + \varepsilon$, where*

$$(A.7) \quad \begin{aligned} \varepsilon &= O\left(\frac{\mu}{|\ell_4|^2 + 1} + \frac{\mu \mathcal{G}}{\chi}\right), \text{ where here and below we choose the upper sign in } \\ &\quad \pm \text{ and } \mp \text{ (the lower sign, resp.) for horizontal outgoing (incoming, resp.)} \\ &\quad \text{asymptotes, when } u > 0 \text{ (} u < 0 \text{ resp.)}. \text{ Then we have the following estimate} \\ &\quad \text{of the derivative of Cartesian coordinates with respect to the Delaunay co-} \\ &\quad \text{ordinates as } \ell \rightarrow \infty \\ \frac{\partial(Q_{\parallel}, Q_{\perp}, P_{\parallel}, P_{\perp})}{\partial(L, \ell, G, g)} &= \mathcal{D} + \varepsilon \cdot \begin{bmatrix} \text{Rot}\left(\frac{\pi}{2}\right) & 0 \\ 0 & \text{Rot}\left(\frac{\pi}{2}\right) \end{bmatrix} \cdot \mathcal{D} + \begin{bmatrix} O(1)_{2 \times 4} \\ 0_{2 \times 4} \end{bmatrix} \\ &\quad + O(\varepsilon^2) \cdot \mathcal{D}, \text{ where } \mathcal{D} = \begin{bmatrix} \pm \frac{2L\ell}{mk} & \pm \frac{L^2}{mk} & \frac{GL^2}{mk(G^2+L^2)} & \mp \frac{GL}{mk} \\ \frac{GL^2\ell}{mk(G^2+L^2)} & 0 & -\frac{L^3\ell}{mk(G^2+L^2)} & \pm \frac{L^2\ell}{mk} \\ \pm \frac{km}{L^2} & -\frac{km}{L\ell^2} & 0 & 0 \\ -\frac{Gkm}{L(G^2+L^2)} & 0 & \frac{km}{(G^2+L^2)} & \mp \frac{km}{L} \end{bmatrix}. \end{aligned}$$

- (b) In the left case, if we assume $g, G = O(\mu\mathcal{G}/\chi)$ and $L = O(1)$, then the estimates of the derivative are obtained by setting $G = O(\mu\mathcal{G}/\chi)$ in the above matrix and choosing $+$ or $-$ according to the sign of u .
- (c) We have

$$\left| \frac{\partial Q}{\partial \ell} \right| = O(1), \quad \left| \frac{\partial Q}{\partial(L, G, g)} \right| = O(\ell), \quad \frac{\partial Q}{\partial g} \cdot Q = 0, \quad \frac{\partial Q}{\partial G} \cdot Q = O_{C^2(L, G, g)}(\ell).$$

Proof. First we drop e in (A.5), since it will contribute $O(1)$ term in (A.7). To obtain the leading term we just need to calculate $\frac{\partial(\tilde{Q}_{\parallel}, \tilde{Q}_{\perp}, P_{\parallel}, P_{\perp})}{\partial(L, \ell, G, g)}(L, \ell, G, \pm \arctan(G/L))$ where \tilde{Q} refers to the RHS of (A.5) with e term discarded. This derivative is obtained by a straightforward calculation using the formulas (A.5), (A.6) with the help of Lemma A.1. The calculations of $\frac{\partial \tilde{Q}}{\partial G}, \frac{\partial \tilde{Q}}{\partial L}$ are presented in details in Lemma A.2 of [DX] and other derivatives are similar. To get the first correction term, i.e. the $O\left(\frac{\mu}{|\ell_4|^2 + 1} + \frac{\mu\mathcal{G}}{\chi}\right)$ part, let $g_0 = \pm \arctan(G/L)$, $\varepsilon = g - g_0$. We use the relations

$$(\tilde{Q}_{\parallel}, \tilde{Q}_{\perp}, P_{\parallel}, P_{\perp})(L, \ell, G, g) = \text{Rot}(g)(\tilde{Q}_{\parallel}, \tilde{Q}_{\perp}, P_{\parallel}, P_{\perp})(L, \ell, G, 0)$$

and

$$\text{Rot}(g_0 + \varepsilon) = \text{Rot}(g_0) + \varepsilon \text{Rot}(\pi/2) \text{Rot}(g_0) + O(\varepsilon^2)$$

and notice that rotation by $\pi/2$ has the effect of interchanging the roles of \parallel and \perp . This gives parts (a) and (b) of the lemma.

Part (c) follows by direct calculation from (A.5) and Lemma A.1. \square

Remark A.1. (1) Part (c) of Lemma A.2 means $\frac{\partial Q}{\partial g}$ is almost parallel to

$\frac{\partial Q}{\partial G}$. This plays an important role in our proof of Lemma 6.2 as well as in [DX]. In fact, in (A.7) the matrix has determinant 1 since it is symplectic. We look at the \mathcal{D} term in (A.7). The discussion remains true when the other terms are included. In \mathcal{D} , the first, second and fourth columns has no obvious linear relations. However, the first and fourth columns has modulus $O(\chi)$ when $|\ell| = O(\chi)$. So the third column must be almost parallel to either the first or fourth column to get determinant one.

- (2) If we look at the rows in \mathcal{D} , similar consideration implies that there should be two rows that are almost parallel. They are the second and fourth rows. This fact plays an important role in the proof of Sublemma 9.1.
- (3) The same argument can be applied to the inverse of the LHS of (A.7). We will see in Lemma A.3 below that the two rows $\frac{\partial g}{\partial(Q, P)}$ and $\frac{\partial G}{\partial(Q, P)}$ are almost parallel, which is used in the proof of Sublemma 9.2 to get the tensor structure.

A.4. The derivative of Delaunay variables with respect to the Cartesian variables. We could have inverted the matrix (A.7) to get the result of this section. However, though the matrix (A.7) is nonsingular, it is close to be singular since we

have some large entries of $O(\chi)$. Therefore we calculate the derivatives $\frac{\partial(L, G, g)}{\partial(Q, P)}$ directly using known identities.

Lemma A.3. *We have the following estimates about the derivatives of Delaunay variables with respect to the Cartesian variables.*

(a) *In the right case, we have as $|\ell| \rightarrow \infty$*

$$(A.8) \quad \begin{aligned} \frac{\partial L}{\partial(Q, P)} &= -\frac{L^3}{mk^2} \left(\frac{kQ}{|Q|^3}, \frac{P}{m} \right), \quad \frac{\partial G}{\partial(Q, P)} = (-P_\perp, P_\parallel, Q_\perp, -Q_\parallel), \\ \frac{\partial g}{\partial(Q, P)} &= \frac{1}{P^2}(0, 0, -P_\perp, P_\parallel) + \\ &\quad \text{sign}(u) \left(\frac{L}{G^2 + L^2} \frac{\partial G}{\partial(Q, P)} - \frac{G}{G^2 + L^2} \frac{\partial L}{\partial(Q, P)} \right) + O(1/\ell^2). \end{aligned}$$

(b) *In the left case, we have the same expressions as part (a) replacing $\text{sign}(u)$ by $+$.*

Proof. From the relation $\frac{mk^2}{2L^2} = \frac{P^2}{2m} - \frac{k}{|Q|}$ we get $\frac{\partial L}{\partial(Q, P)} = -\frac{L^3}{mk^2} \left(\frac{kQ}{|Q|^3}, \frac{P}{m} \right)$. We also have

$$G = P \times Q, \quad \frac{\partial G}{\partial(Q, P)} = (-P_\perp, P_\parallel, Q_\perp, -Q_\parallel).$$

To get the derivative $\frac{\partial g}{\partial(Q, P)}$, consider the right case first. We take quotient P_\perp/P_\parallel in (A.5), then apply the formula $\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$ to get in the right case

$$(A.9) \quad g = \arctan \frac{P_\perp}{P_\parallel} + \text{sign}(u) \arctan \frac{G}{L} - e^{-2|u|} E(G/L, g) + o(e^{-2|u|}) \pmod{\pi}.$$

where E is a smooth function. The sign is chosen in such a way that g is close to $\text{sign}(u) \arctan \frac{G}{L}$ as analyzed after (A.4). Hence

$$(A.10) \quad \begin{aligned} \partial g &= \frac{P_\parallel \partial P_\perp - P_\perp \partial P_\parallel}{P_\parallel^2 + P_\perp^2} + \text{sign}(u) \frac{L \partial G - G \partial L}{G^2 + L^2} + O\left(\frac{1}{\ell^2}\right) \\ &= \frac{1}{P^2}(0, 0, -P_\perp, P_\parallel) + \text{sign}(u) \frac{L \partial G - G \partial L}{G^2 + L^2} + O\left(\frac{1}{\ell^2}\right). \end{aligned}$$

In the left case, we use (A.6) to see that g satisfies (A.9) with $\text{sign}(u)$ replaced by $+$. \square

A.5. Second order derivatives. The following estimates of the second order derivatives are used in integrating the variational equation.

Lemma A.4. *Assume that $|G| \leq K$, $1/K \leq L \leq K$.*

(a) *We have*

$$\frac{\partial^2 Q}{\partial g^2} = -Q, \quad \frac{\partial^2 Q}{\partial g \partial G} \perp \frac{\partial Q}{\partial G}, \quad \left(\frac{\partial}{\partial G}, \frac{\partial}{\partial g} \right) \left(\frac{\partial |Q|^2}{\partial g} \right) = (0, 0),$$

$$\frac{\partial^2 Q}{\partial G^2} = O(\ell), \quad \frac{\partial^2 Q}{\partial L^2} = O(\ell).$$

(b) Under the conditions of Lemma A.2(a) we have we have

$$\begin{aligned} \frac{\partial^2 Q}{\partial G^2} &= \frac{L^2}{(L^2 + G^2)^{3/2}} (L \cosh u, -2G \sinh u) + O(\mu \mathcal{G}), \\ \frac{\partial^2 Q}{\partial g \partial G} &= \left(-\frac{L^2 \sinh u}{\sqrt{L^2 + G^2}}, 0 \right) + O(\mu \mathcal{G}), \\ \frac{\partial^2 Q}{\partial g \partial L} &= \left(\frac{GL \sinh u}{\sqrt{L^2 + G^2}}, -2\sqrt{L^2 + G^2} \cosh u \right) + O(\mu \mathcal{G}), \\ \frac{\partial^2 Q}{\partial G \partial L} &= \frac{L}{(L^2 + G^2)^{3/2}} (-LG \cosh u, (L^2 + 3G^2) \sinh u) + O(\mu \mathcal{G}). \end{aligned}$$

(c) Under the conditions of Lemma A.2(b) we have

$$\begin{aligned} \frac{\partial^2 Q}{\partial G^2} &= -\cosh u(1, 0) + O(\mu \mathcal{G}), & \frac{\partial^2 Q}{\partial g \partial G} &= -L \sinh u(1, 0) + O(\mu \mathcal{G}), \\ \frac{\partial^2 Q}{\partial g \partial L} &= L \sinh u(0, 2) + O(\mu \mathcal{G}), & \frac{\partial^2 Q}{\partial G \partial L} &= \cosh u(0, 1) + O(\mu \mathcal{G}). \end{aligned}$$

Proof. Part (a) is trivial. The proof proceeds by formal calculations. Let us consider $\frac{\partial^2 Q}{\partial G^2}$ in part (b) for example. Let us first assume $g = \pm \arctan \frac{G}{L}$ with the correct choice of sign according to Lemma A.2. We get

$$\frac{\partial^2 Q}{\partial G^2} = \frac{L^2}{(L^2 + G^2)^{3/2}} (L \cosh u, -2G \sinh u) + O(1).$$

Details of this calculation can be find in Lemma A.3 in the Appendix A of [DX].

Next, we introduce the $O\left(\frac{\mu}{|\ell_4|^2 + 1} + \frac{\mu \mathcal{G}}{\chi}\right)$ perturbation to g . We use the same argument as in the proof of Lemma A.2 to get the first order correction is given by

$$\text{Rot}(\pi/2) \frac{L^2}{(L^2 + G^2)^{3/2}} (L \cosh u, -2G \sinh u) \cdot \mu \mathcal{G} / \ell = O(\mu \mathcal{G}).$$

All the other second derivatives are done similarly with the help of the detailed calculation in Lemma A.3 in the Appendix A of [DX]. \square

APPENDIX B. GERVER'S MECHANISM

B.1. Gerver's result in [G2]. We summarize the result of [G2] in the following table. Recall that the Gerver scenario deals with the limiting case $\chi \rightarrow \infty, \mu \rightarrow 0$. Accordingly Q_1 disappears at infinity and there is no interaction between Q_3 and Q_4 . Hence both particles perform Kepler motions. The shape of each Kepler orbit is characterized by energy, angular momentum and the argument of periapsis. In Gerver's scenario, the incoming and outgoing asymptotes of the hyperbola are always horizontal and the semimajor of the ellipse is always vertical. So we only need to describe on the energy and angular momentum.

	1st collision	@ $(-\varepsilon_0\varepsilon_1, \varepsilon_0 + \varepsilon_1)$	2nd collision	@ $(\varepsilon_0^2, 0)$
	Q_3	Q_4	Q_3	Q_4
energy	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} \rightarrow -\frac{\varepsilon_1^2}{2\varepsilon_0^2}$	$\frac{1}{2} \rightarrow \frac{\varepsilon_1^2}{2\varepsilon_0^2}$
angular momentum	$\varepsilon_1 \rightarrow -\varepsilon_0$	$p_1 \rightarrow -p_2$	$-\varepsilon_0$	$\sqrt{2}\varepsilon_0$
eccentricity	$\varepsilon_0 \rightarrow \varepsilon_1$		$\varepsilon_1 \rightarrow \varepsilon_0$	
semimajor	1	-1	$1 \rightarrow \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^2$	$1 \rightarrow -\frac{\varepsilon_1^2}{\varepsilon_0^2}$
semiminor	$\varepsilon_1 \rightarrow \varepsilon_0$	$p_1 \rightarrow p_2$	$\varepsilon_0 \rightarrow \frac{\varepsilon_0^2}{\varepsilon_1}$	$\sqrt{2}\varepsilon_0 \rightarrow \sqrt{2}\varepsilon_1$

Here

$$p_{1,2} = \frac{-Y \pm \sqrt{Y^2 + 4(X+R)}}{2}, \quad R = \sqrt{X^2 + Y^2}.$$

and (X, Y) stands for the point where collision occurs (the parenthesis after @ in the table). We will call the two points the Gerver's collision points.

In the above table ε_0 is a free parameter and $\varepsilon_1 = \sqrt{1 - \varepsilon_0^2}$.

At the collision points, the velocities of the particles are the following.

For the first collision,

$$v_3^- = \left(\frac{-\varepsilon_1^2}{\varepsilon_0\varepsilon_1 + 1}, \frac{-\varepsilon_0}{\varepsilon_0\varepsilon_1 + 1} \right), \quad v_4^- = \left(1 - \frac{Y}{Rp_1}, \frac{1}{Rp_1} \right).$$

$$v_3^+ = \left(\frac{\varepsilon_0^2}{\varepsilon_0\varepsilon_1 + 1}, \frac{\varepsilon_1}{\varepsilon_0\varepsilon_1 + 1} \right), \quad v_4^+ = \left(-1 + \frac{Y}{Rp_2}, -\frac{1}{Rp_2} \right).$$

For the second collision,

$$v_3^- = \left(\frac{-\varepsilon_1}{\varepsilon_0}, \frac{-1}{\varepsilon_0} \right), \quad v_4^- = \left(1, \frac{\sqrt{2}}{\varepsilon_0} \right), \quad v_3^+ = \left(1, \frac{-1}{\varepsilon_0} \right), \quad v_4^+ = \left(\frac{-\varepsilon_1}{\varepsilon_0}, \frac{\sqrt{2}}{\varepsilon_0} \right).$$

B.2. Numerical information for a particularly chosen $\varepsilon_0 = 1/2$. For the first collision $e_3 : \frac{1}{2} \rightarrow \frac{\sqrt{3}}{2}$.

We want to figure out the Delaunay coordinates (L, u, G, g) for both Q_3 and Q_4 . (Here we replace ℓ by u for convenience.) The first collision point is

$$(X, Y) = (-\varepsilon_0\varepsilon_1, \varepsilon_0 + \varepsilon_1) = \left(-\frac{\sqrt{3}}{4}, \frac{1 + \sqrt{3}}{2} \right).$$

Before collision

$$(L, u, G, g)_3^- = \left(1, -\frac{5\pi}{6}, \frac{\sqrt{3}}{2}, \pi/2 \right), \quad (L, u, G, g)_4^- = (1, 1.40034, p_1, -\arctan p_1),$$

$$v_3^- = \left(\frac{-3}{\sqrt{3} + 4}, \frac{-2}{\sqrt{3} + 4} \right) \simeq -(0.523, 0.349),$$

$$v_4^- = \left(1 - \frac{2(1 + \sqrt{3})}{(4 + \sqrt{3})p_1}, \frac{4}{(4 + \sqrt{3})p_1} \right) \simeq (-0.805, 1.322),$$

where

$$p_1 = \frac{-Y + \sqrt{Y^2 + 4(X+R)}}{2} = \frac{-(\varepsilon_0 + \varepsilon_1) + \sqrt{5 + 2\varepsilon_0\varepsilon_1}}{2} = 0.52798125.$$

After collision

$$\begin{aligned}(L, u, G, g)_3^+ &= \left(1, \frac{2\pi}{3}, -\frac{1}{2}, \pi/2\right), \quad (L, u, G, g)_4^+ = (1, 0.515747, -p_2, -\arctan p_2), \\ v_3^+ &= \left(\frac{1}{\sqrt{3}+4}, \frac{2\sqrt{3}}{\sqrt{3}+4}\right) \simeq (0.174, 0.604), \\ v_4^+ &= \left(-1 + \frac{2(1+\sqrt{3})}{(4+\sqrt{3})p_2}, -\frac{4}{(4+\sqrt{3})p_2}\right) \simeq (-1.503, 0.368)\end{aligned}$$

where

$$p_2 = \frac{-Y - \sqrt{Y^2 + 4(X+R)}}{2} = \frac{-(\varepsilon_0 + \varepsilon_1) - \sqrt{5 + 2\varepsilon_0\varepsilon_1}}{2} = -1.894006654.$$

For the second collision $e_3 : \frac{\sqrt{3}}{2} \rightarrow \frac{1}{2}$.

The collision point is $(X, Y) = (\varepsilon_0^2, 0) = \left(\frac{1}{4}, 0\right)$.

Before collision

$$\begin{aligned}(L, u, G, g)_3^- &= \left(1, -\frac{\pi}{6}, -\frac{1}{2}, \pi/2\right), \quad (L, u, G, g)_4^- = \left(1, 0.20273, \sqrt{2}/2, -\arctan \frac{\sqrt{2}}{2}\right), \\ v_3^- &= (-\sqrt{3}, -2), \quad v_4^- = (1, 2\sqrt{2}).\end{aligned}$$

After collision

$$\begin{aligned}(L, u, G, g)_3^+ &= \left(\frac{1}{\sqrt{3}}, \frac{\pi}{3}, -\frac{1}{2}, -\frac{\pi}{2}\right), \quad (L, u, G, g)_4^+ = \left(\frac{1}{\sqrt{3}}, -0.45815, \frac{\sqrt{2}}{2}, \arctan \frac{\sqrt{2}}{2}\right), \\ v_3^+ &= (1, -2), \quad v_4^+ = (-\sqrt{3}, 2\sqrt{2}).\end{aligned}$$

B.3. Control the shape of the ellipse. As it was mentioned before Lemma 2.1 was stated by Gerver in [G2]. There is a detailed proof of part (a) of our Lemma 2.1 in [G2]. However since no details of the proof of part (b) were given in [G2] we go other main steps here for the reader's convenience even though computations are quite straightforward.

Proof of Lemma 2.1. Recall that Gerver's map depends on a free parameter e_4 (or equivalently G_4). In the computations below however it is more convenient to use the polar angle ψ of the intersection point as the free parameter. It is easy to see that as G_4 changes from large negative to large positive value the point of intersection covers the whole orbit of Q_3 so it can be used as the free parameter. Our goal is to show that by changing the angles ψ_1 and ψ_2 of the first and second collision we can prescribe the values of \bar{e}_3 and \bar{g}_3 arbitrarily. Due to the Implicit Function Theorem it suffices to show that

$$\det \begin{bmatrix} \frac{\partial \bar{e}_3}{\partial \psi_1} & \frac{\partial \bar{g}_3}{\partial \psi_1} \\ \frac{\partial \bar{e}_3}{\partial \psi_2} & \frac{\partial \bar{g}_3}{\partial \psi_2} \end{bmatrix} \neq 0.$$

To this end we use the following set of equations

$$(B.1) \quad G_3^+ + G_4^+ = G_3^- + G_4^-,$$

$$(B.2) \quad \frac{e_3^+}{G_3^+} \cos(\psi + g_3^+) + \frac{e_4^+}{G_4^+} \cos(\psi - g_4^-) = \frac{e_3^-}{G_3^-} \cos(\psi + g_3^-) + \frac{e_4^-}{G_4^-} \cos(\psi - g_4^-),$$

$$(B.3) \quad \frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi + g_3^+)} = \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi + g_3^-)},$$

$$(B.4) \quad \frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi + g_3^+)} = \frac{(G_4^+)^2}{1 - e_4^+ \sin(\psi - g_4^+)},$$

$$(B.5) \quad g_4^+ = \arctan \frac{G_4^+}{L_4^+}.$$

Here e_3, e_4 and L_4 are functions of the other variables according to the formulas of Appendix A.

(B.1)–(B.5) are obtained as follows. (B.1) is the angular momentum conservation, (B.3) means that the position of Q_3 does not change during the collision, (B.4) means that Q_3 and Q_4 are at the same point immediately after the collision and (B.5) says that after the collision the outgoing asymptote of Q_4 is horizontal.

It remains to derive (B.2). Represent the position vector as $\vec{r} = r\hat{e}_r$. Then the velocity is $\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi$. The momentum conservation gives

$$(\dot{\vec{r}}_3)^- + (\dot{\vec{r}}_4)^- = (\dot{\vec{r}}_3)^+ + (\dot{\vec{r}}_4)^+.$$

Taking the angular component of the velocity we get

$$(B.6) \quad r_3^- \dot{\psi}_3^- + r_4^- \dot{\psi}_4^- = r_3^+ \dot{\psi}_3^+ + r_4^+ \dot{\psi}_4^+.$$

In our notation the polar representation of the ellipse takes form $r = \frac{G^2}{1 - e \sin(\psi + g)}$. Differentiating this equation we obtain the following relation for the radial component of the Kepler motion

$$\dot{r} = \frac{G^2}{(1 - e \sin(\psi + g))^2} e \cos(\psi + g) \dot{\psi} = \frac{r^2}{G^2} e \cos(\psi + g) \frac{G}{r^2} = \frac{e}{G} \cos(\psi + g).$$

Plugging this into (B.6) we obtain (B.2).

We can write (B.1)–(B.5) in the form

$$\mathbb{F}(Z^-, \tilde{Z}, Z^+) = 0$$

where $Z^- = (E_3^-, G_3^-, g_3^-, \psi)$, $Z^+ = (E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)$, and $\tilde{Z} = (G_4^-, g_4^-)$ are considered as functions Z^- .

By the Implicit Function Theorem we have

$$\frac{\partial Z^+}{\partial Z^-} = - \left(\frac{\partial \mathbb{F}}{\partial Z^+} \right)^{-1} \left(\frac{\partial \mathbb{F}}{\partial Z^-} + \frac{\partial \mathbb{F}}{\partial \tilde{Z}} \frac{\partial \tilde{Z}}{\partial Z^-} \right).$$

Thus to complete the computation we need to know $\frac{\partial \tilde{Z}}{\partial Z^-}$. In order to compute this expression we use the equations

$$(B.7) \quad g_4^- = -\arctan \frac{G_4^-}{L_4^-}$$

which means that the incoming asymptote of Q_4 is horizontal and

$$(B.8) \quad \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi + g_3^-)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi - g_4^-)},$$

which means that Q_3 and Q_4 are at the same place immediately before the collision. Writing these equations as $\mathbb{I}(Z^-, \tilde{Z}) = 0$ we get by the Implicit Function Theorem

$$\frac{\partial \tilde{Z}}{\partial Z^-} = - \left(\frac{\partial \mathbb{I}}{\partial \tilde{Z}} \right)^{-1} \frac{\partial \mathbb{I}}{\partial Z^-}$$

so that the required derivative equals to

$$(B.9) \quad \frac{\partial Z^+}{\partial Z^-} = - \left(\frac{\partial \mathbb{F}}{\partial Z^+} \right)^{-1} \left(\frac{\partial \mathbb{F}}{\partial Z^-} - \frac{\partial \mathbb{F}}{\partial \tilde{Z}} \left(\frac{\partial \mathbb{I}}{\partial \tilde{Z}} \right)^{-1} \frac{\partial \mathbb{I}}{\partial Z^-} \right).$$

Combining (B.9) with the formula

$$de_3 = - \frac{2G_3 E_3 dG_3 + G_3^2 dE_3}{\sqrt{1 - 2G_3^2 E_3}}$$

which follows from the relation $e_3 = \sqrt{1 - 2G_3^2 E_3}$ we obtain the two entries

$$\frac{\partial \bar{e}_3}{\partial \psi_2} = -0.158494 \text{ and } \frac{\partial \bar{g}_3}{\partial \psi_2} = 0.369599.$$

The meanings of these two entries are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the *second* collision.

We need more work to figure out the two entries $\frac{\partial \bar{e}_3}{\partial \psi_1}$ and $\frac{\partial \bar{g}_3}{\partial \psi_1}$, which are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the *first* collision. We describe the computation of the first entry, the second one is similar. We use the relation

$$\frac{\partial \bar{e}_3}{\partial \psi_1} = \frac{\partial \bar{e}_3}{\partial \bar{E}_3^+} \frac{\partial \bar{E}_3^+}{\partial \psi_1} + \frac{\partial \bar{e}_3}{\partial \bar{G}_3^+} \frac{\partial \bar{G}_3^+}{\partial \psi_1} + \frac{\partial \bar{e}_3}{\partial \bar{g}_3^+} \frac{\partial \bar{g}_3^+}{\partial \psi_1}.$$

Now $\left(\frac{\partial \bar{E}_3^+}{\partial \psi_1}, \frac{\partial \bar{G}_3^+}{\partial \psi_1}, \frac{\partial \bar{g}_3^+}{\partial \psi_1} \right)$ is computed using (B.9) and the data for the first collision. Noticing that the quantities E_3, G_3, g_3 after the first collision are the same as those before the second collision, we replace $\left(\frac{\partial \bar{e}_3}{\partial \bar{E}_3^+}, \frac{\partial \bar{e}_3}{\partial \bar{G}_3^+}, \frac{\partial \bar{e}_3}{\partial \bar{g}_3^+} \right)$ by $\left(\frac{\partial \bar{e}_3}{\partial \bar{E}_3^-}, \frac{\partial \bar{e}_3}{\partial \bar{G}_3^-}, \frac{\partial \bar{e}_3}{\partial \bar{g}_3^-} \right)$ and compute it using (B.9) and the data for the second collision. It turns out that the resulting matrix is

$$\begin{bmatrix} \frac{\partial \bar{e}_3}{\partial \psi_1} & \frac{\partial \bar{g}_3}{\partial \psi_1} \\ \frac{\partial \bar{e}_3}{\partial \psi_2} & \frac{\partial \bar{g}_3}{\partial \psi_2} \end{bmatrix} = \begin{bmatrix} 0.620725 & 2.9253 \\ -0.158494 & 0 \end{bmatrix},$$

which is obviously nondegenerate. \square

APPENDIX C. \mathcal{C}^1 CONTROL OF THE GLOBAL MAP, PROOF OF LEMMA 3.2

In this Appendix, we derive Lemma 3.2 from Proposition 5.1. We split the proof into six steps. In Step 0, we make preparations by defining some auxiliary vectors and the matrix S and simplify matrices (I)–(V) from Proposition 5.1. Then we start multiplying the matrices. The calculation is done symmetrically. In Step 1, we consider the multiplication $(IV)(III)(II)$. We decompose this product as three summands, which will give rise to the χ^2 part, χ part and $\mu\chi$ part. In the following steps we analyze the three summands one by one. In Step 2, we prove the χ^2 part of the Lemma 3.2. In Step 3, we show that one summand in Step 1 is of order $\mu\chi$. In Step 4, we prove the χ part of the Lemma 3.2. As a matter of fact, the χ^2 and χ parts of the global map are the same as in [DX]. The $\mu\chi$ perturbation comes mainly from the Q_1 . In Step 5, we summarize the calculation to complete the proof of Lemma 3.2.

In the proof, we need some sublemmas and its corollaries, which condense the brute force calculations. The first of them is easy. The remaining sublemmas are proven with the help of computer. In almost all the places, what we encounter is multiplication of a vector with a matrix, each of which has many almost vanishing entries. So the calculation gets simplified significantly. For the estimates of vectors appearing in the sub lemmas, we need only the estimate of their norms (instead of the vector form estimates) and the corresponding corollaries to complete the proof. Finally, we explain in Remark C.1 where the main terms in Lemma 3.2 come from. Thus the computer is only used to show that the remaining terms make small ($O(\mu\chi)$) correction. In almost all the places, it is enough to use \lesssim or O to do the estimates. However, we need the exact information of the vectors $l_i, u_{iii}, u_{iii'}$ and the blocks $(N_1)_{44}, (M)_{44}, (N_5)_{44}$.

STEP 0: *Preparations. Definitions of auxiliary vectors and simplification of the five matrices.*

We define some auxiliary new vectors. Recall that in the paragraph before Proposition 5.1, we introduced a convention to use **bold** font to indicate that the estimate of the corresponding entry is actually \sim , not only \lesssim .

Below we use the following notational convention to make it easier to the reader to keep track of the computations. A vector with *tilde*, *hat*, *bar* means a $O(1/\chi), O(\mu), O(1)$ perturbation to the vector respectively.

Definition C.1. We define the following list of vectors and the matrix S .

- $\tilde{u} := Mu = u + O\left(\frac{1}{\chi}\right) \lesssim \left(\frac{1}{\chi^2}, \mathbf{1}, \frac{1}{\chi^2}, \frac{1}{\chi^2}; \mu, \frac{\mu\mathcal{G}}{\chi}, \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}; \frac{\mu\mathcal{G}}{\chi^2}, \frac{\mu\mathcal{G}}{\chi^2}\right)^T$,
 $\tilde{l} := lM = l + O\left(\frac{1}{\chi}\right) \lesssim \left(\mathbf{1}, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi^2}; \frac{1}{\mu\chi^2}, \frac{\mathcal{G}}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi}; \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi}\right)$,
- $\hat{u} := N_5 u = u + O(\mu) \lesssim \left(\mu, 1, \mu, \mu; \mu, \frac{\mu\mathcal{G}}{\chi}, \frac{1}{\mu\chi^2}, \frac{\mu}{\chi^2}; \mu, \mu\right)$,
 $\hat{l} := lN_1 = l + O(\mu) \lesssim \left(1, \mu, \mu, \mu; \frac{1}{\mu\chi^2}, \frac{\mu^2\mathcal{G}^2}{\chi^3}, \mu, \frac{\mu\mathcal{G}}{\chi}; \mu, \mu\right)$,
- $\delta u := AL \cdot R^{-1}u_i \sim AR \cdot L^{-1}u_{i'} \lesssim \left(0, 0, 0, 0; 0, \mu, 0, \frac{1}{\chi}; 0, \frac{1}{\chi}\right)^T = O(\mu)$,
 $\delta l := l_{iii} L \cdot R^{-1}C \sim l_{iii'} R \cdot L^{-1}C \lesssim \left(\frac{\mathcal{G}}{\chi}, \frac{\mathcal{G}}{\chi^4}, \frac{\mathcal{G}}{\chi^4}, \frac{\mathcal{G}}{\chi^4}; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \mu; \frac{1}{\chi}, \frac{1}{\chi}\right) = O(\mu)$,
- $\hat{u}_{iii} := u_{iii} + \delta u \lesssim \left(0, 0, 0, 0; 0, \mu, 0, \frac{1}{\chi}; 1, 1\right)^T$,
 $\hat{l}_i := l_i + \delta l \lesssim \left(1, \frac{1}{\chi^3}, \frac{1}{\chi^3}, \frac{1}{\chi^3}; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \mu, \mu; 1, 1\right)$,
 $\hat{u}_{iii'} := u_{iii'} + \delta u \lesssim \left(0, 0, 0, 0; 0, \mu, 0, \frac{1}{\chi}; 1, 1\right)^T$,
 $\hat{l}_{i'} := l_{i'} + \delta l \lesssim \left(\frac{\mathcal{G}}{\chi}, \frac{\mathcal{G}}{\chi^4}, \frac{\mathcal{G}}{\chi^4}, \frac{\mathcal{G}}{\chi^4}; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \mu; 1, 1\right)$,

$$AL \cdot R^{-1}C = S_2, \quad AR \cdot L^{-1}C = S_4, \quad \text{where } S_2, S_4 \lesssim S :=$$

$$\begin{bmatrix} \text{Id}_{4 \times 4} & & & & & & & \\ & 0_{1 \times 3} & & & & & & \\ & 0 & 0_{1 \times 3} & & & & & \\ & O(1) & O\left(\frac{1}{\chi^3}\right)_{1 \times 3} & & & & & \\ & 0 & 0_{1 \times 3} & & & & & \\ & 0 & 0_{1 \times 3} & & & & & \\ & 1 + O(\mu) & O\left(\frac{1}{\chi^3}\right)_{1 \times 3} & & & & & \\ & & & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 2} \\ & & & 1 + O(\mu) & 0 & 0 & 0 & 0_{1 \times 2} \\ & & & 0 & 1 + O(\mu) & 0 & 0 & O(\mu)_{1 \times 2} \\ & & & O\left(\frac{1}{\mu\chi^2}\right) & O\left(\frac{\mathcal{G}}{\chi^3}\right) & 1 + O(\mu) & O\left(\frac{\mu\mathcal{G}}{\chi}\right) & O\left(\frac{\mu\mathcal{G}}{\chi}\right)_{1 \times 2} \\ & & & 0 & 0 & 0 & 1 + O(\mu) & 0_{1 \times 2} \\ & & & 0 & 0 & 0 & 0 & 0_{1 \times 2} \\ & & & O\left(\frac{1}{\mu\chi^2}\right) & O\left(\frac{1}{\chi^2}\right) & O(\mu) & O(\mu) & O\left(\frac{\mu\mathcal{G}}{\chi}\right)_{1 \times 2} \end{bmatrix}.$$

Sublemma C.1. (1) $l \cdot u, \tilde{l} \cdot u, l \cdot \tilde{u} \lesssim \frac{1}{\chi^2}$,

$$(2) \ l_{iii} \cdot L \cdot R^{-1}u_i = \frac{m_-}{\chi}, \quad l_{iii'} \cdot R \cdot L^{-1}u_{i'} = -\frac{m_+}{\chi}.$$

Proof. All of these estimates are straightforward calculation using Proposition 5.1 and Definition C.1. Item (2) is exact. It uses Proposition 5.1(b2). \square

Using the Definition C.1 and Sublemma C.1, we simplify the five matrices into sums as follows. Notice that the factors $(\text{Id}_{10} + u_1^i \otimes l_1^i)$ in (I) and $(\text{Id}_{10} + u_5^f \otimes l_5^f)$ in (V) are both $O(1)$, so we do not include them in the following calculation until the

final step for simplicity. We shall write (\bar{I}) and (\bar{V}) for the modified matrices.

$$\begin{aligned}
(C.1) \quad (III) &\lesssim (\text{Id}_{10} + \chi u \otimes l)M(\text{Id}_{10} + \chi u \otimes l') = (\text{Id}_{10} + \chi u \otimes l)(M + \chi M u \otimes l') \\
&= (\text{Id}_{10} + \chi u \otimes l)(M + \chi \tilde{u} \otimes l') = (M + \chi \tilde{u} \otimes l') + \chi u \otimes l(M + \chi \tilde{u} \otimes l') \\
&\sim M + \chi \tilde{u} \otimes l' + \chi u \otimes \tilde{l}, \\
(II) &= (\chi u_{iii} \otimes l_{iii} + A)L \cdot R^{-1}(\chi u_i \otimes l_i + C) \\
&= \chi^2 u_{iii} \otimes l_{iii} L \cdot R^{-1} u_i \otimes l_i + \chi \delta u \otimes l_i + \chi u_{iii} \otimes \delta l + S \\
&\sim \chi \hat{u}_{iii} \otimes \hat{l}_i - \chi \delta u \otimes \delta l + S, \\
(IV) &\sim \chi \hat{u}_{iii'} \otimes \hat{l}_{i'} - \chi \delta u \otimes \delta l' + S, \\
(\bar{I}) &\lesssim (\text{Id}_{10} + \chi u \otimes l)N_1 = N_1 + \chi u \otimes l N_1 = N_1 + \chi u \otimes \hat{l}, \\
(\bar{V}) &\lesssim N_5(\text{Id}_{10} + \chi u \otimes l') = N_5 + N_5 \chi u \otimes l' = N_5 + \chi \hat{u} \otimes l',
\end{aligned}$$

where for (III) , we used that $l(M + \chi \tilde{u} \otimes l') \lesssim \tilde{l} + \frac{1}{\chi} l' \sim \tilde{l}$ by Definition C.1(first bullet point) and Sublemma C.1(1).

STEP 1: *Decomposing $(IV)(III)(II)$ into three summands.*

We start with an auxillary estimate.

Sublemma C.2. *We have the following estimates as $1/\chi \ll \mu \rightarrow 0$.*

$$\begin{aligned}
(1) \quad (III)\hat{u}_{iii} &\lesssim \left(\frac{\mu \mathcal{G}}{\chi^2}, \mu \mathcal{G}, \frac{\mu \mathcal{G}}{\chi^2}, \frac{\mu \mathcal{G}}{\chi^2}; \mu^2 \mathcal{G}, \mu, \frac{\mu \mathcal{G}}{\chi^2}, \frac{1}{\chi}; \mathbf{1}, \mathbf{1} \right)^T = O(\mu \mathcal{G}), \\
\hat{l}_{i'}(III) &\lesssim \left(1, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi^2}; \frac{\mu \mathcal{G}}{\chi^2}, \frac{1}{\chi}, \mu, \mu; \mathbf{1}, \mathbf{1} \right) = O(1), \\
(2) \quad \hat{l}_{i'}(III)\hat{u}_{iii} &\rightarrow \frac{-2}{\tilde{L}_{4,j}^2}.
\end{aligned}$$

Corollary C.1. (1) $\delta l(III)\hat{u}_{iii} \lesssim \frac{1}{\chi},$

(2) $\hat{l}_{i'}(III)\delta u \lesssim \frac{1}{\chi}.$

Proof. All of these is done by straightforward calculation using the information obtained in Proposition 5.1 together with the calculation of (III) in (C.1). The 1 entries in item (1) are actually $(M)_{44}(u_{iii}(9), u_{iii}(10))$ and $(l_{i'}(9), l_{i'}(10))(M)_{44}$ up to a $O(\mu)$ error. Item (2) is in fact $(l_{i'}(9), l_{i'}(10))(M)_{44}(u_{iii}(9), u_{iii}(10))$ up to a $O(\mu)$ error. These terms can be calculated explicitly using part (b1), (b2), (b3) of Proposition 5.1. \square

Then we consider

$$\begin{aligned}
(C.2) \quad (IV)(III)(II) &\sim (\chi \hat{u}_{iii'} \otimes \hat{l}_{i'} - \chi \delta u \otimes \delta l + S)(III)(\chi \hat{u}_{iii} \otimes \hat{l}_i - \chi \delta u \otimes \delta l + S) \\
&= \chi^2 \hat{u}_{iii'} \otimes \hat{l}_{i'}(III)\hat{u}_{iii} \otimes \hat{l}_i + (-\chi \delta u \otimes \delta l + S)(III)(\chi \hat{u}_{iii} \otimes \hat{l}_i) \\
&\quad + (\chi \hat{u}_{iii'} \otimes \hat{l}_{i'})(III)(-\chi \delta u \otimes \delta l + S) + (-\chi \delta u \otimes \delta l + S)(III)(-\chi \delta u \otimes \delta l + S).
\end{aligned}$$

Define

$$v = \hat{l}_{i'}(III)(-\chi \delta u \otimes \delta l + S), \quad v' = (-\chi \delta u \otimes \delta l + S)(III)\hat{u}_{iii}.$$

Both are of order 1 by Corollary C.1 and Sublemma C.2 (1). From Sublemma C.2(2) we get

$$\begin{aligned} (C.2) &\sim \chi^2 \hat{u}_{iii'} \otimes \hat{l}_i + \chi v' \otimes \hat{l}_i + \chi \hat{u}_{iii'} \otimes v + (\chi \delta u \otimes \delta l - S)(III)(\chi \delta u \otimes \delta l - S) \\ &= \chi^2 \left(\hat{u}_{iii'} + \frac{1}{\chi} v' \right) \otimes \left(\hat{l}_i + \frac{1}{\chi} v \right) - v' \otimes v + (\chi \delta u \otimes \delta l - S)(III)(\chi \delta u \otimes \delta l - S) \end{aligned}$$

$$(C.3) \quad := \chi^2 \tilde{u}_{iii'} \otimes \tilde{l}_i - v' \otimes v + (\chi \delta u \otimes \delta l - S)(III)(\chi \delta u \otimes \delta l - S),$$

where we have defined

$$\tilde{u}_{iii'} = u_{iii'} + \delta u + \frac{1}{\chi} v', \quad \tilde{l}_i = l_i + \delta l + \frac{1}{\chi} v.$$

It is important to stress that the coefficient of χ^2 term is nonzero. Then we consider $(V)(IV)(III)(II)(I) = (V)(C.3)(I)$.

In the following, we are going to show that the $\chi^2(V)\tilde{u}_{iii'} \otimes \tilde{l}_i(I)$ gives rise to the χ^2 part of the main lemma 3.2. The $(V)v' \otimes v(I)$ part will be absorbed into $O(\mu\chi)$ part. The last summand in (C.3) will give rise to $O(\chi)$ part together with a perturbation of order $O(\mu\chi)$, where the $O(\chi)$ part comes from $(V)S(III)S(I)$.

STEP 2: *The first summand in (C.3) gives the $O(\chi^2)$ contributions in Lemma 3.2.*

The following sub lemma is needed for this step.

Sublemma C.3. (1) $l' \cdot \hat{u}_{iii'} \lesssim \frac{\mu\mathcal{G}}{\chi}$, $\hat{l}_i \cdot u \lesssim \frac{\mu\mathcal{G}}{\chi^2}$.
 (2) $l' \cdot v' \lesssim \mu$, $v \cdot u \lesssim \mu$.

We consider first the term $(\bar{V})(\chi^2 \tilde{u}_{iii'} \otimes \tilde{l}_i)(\bar{I})$. We keep in mind that $N_1, N_5 = O(\mu\chi)$. Define

$$\begin{aligned} (C.4) \quad \bar{u}' &:= (\bar{V})\tilde{u}_{iii'} = N_5 \tilde{u}_{iii'} + \chi \hat{u} \otimes l' \cdot \tilde{u}_{iii'} = (N_5 \hat{u}_{iii'} + O(\mu)) + \hat{u} (\chi l' \cdot \hat{u}_{iii'} + l' \cdot v') \\ &= N_5 \hat{u}_{iii'} + O(\mu\mathcal{G})\hat{u} + O(\mu), \\ \bar{l}' &:= \tilde{l}_i(\bar{I}) = \tilde{l}_i N_1 + \chi \tilde{l}_i \cdot u \otimes \hat{l} = (\hat{l}_i N_1 + O(\mu)) + (\chi \hat{l}_i \cdot u + v \cdot u) \hat{l} = \hat{l}_i N_1 + O(\mu). \end{aligned}$$

We will analyze \bar{u}' and \bar{l}' in more details in the final step.

STEP 3: *The second summand in (C.3) is $O(\mu\chi)$.*

The following sub lemma is needed in this step.

Sublemma C.4. *We have the following estimates.*

$$\begin{aligned} (1) \quad N_5 \delta u &\lesssim \left(\frac{\mu^2 \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi}, \frac{\mu^2 \mathcal{G}}{\chi}, \frac{\mu^2 \mathcal{G}}{\chi}; \frac{\mu^2 \mathcal{G}}{\chi}, \mu, \frac{\mu^2 \mathcal{G}}{\chi^2}, \frac{1}{\chi}; \frac{1}{\chi}, \frac{1}{\chi}, \frac{1}{\chi} \right)^T = O(\mu), \\ \delta l N_1 &\lesssim \left(\frac{\mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi}, \frac{\mu \mathcal{G}}{\chi}; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \frac{\mu \mathcal{G}}{\chi}, \mu; \frac{1}{\chi}, \frac{1}{\chi} \right) = O(\mu), \\ (2) \quad l' \cdot \delta u &\lesssim \frac{\mu \mathcal{G}}{\chi^2}, \quad \delta l \cdot u \lesssim \frac{\mu \mathcal{G}}{\chi^2}. \end{aligned}$$

Before considering $(\bar{V})v' \otimes v(\bar{I})$, we perform the following calculation.

(C.5)

$$\begin{aligned} (\bar{V})\chi\delta u \otimes \delta l &= (N_5 + \chi\hat{u} \otimes l')\chi\delta u \otimes \delta l = \chi(N_5\delta u + \chi\hat{u} \otimes l' \cdot \delta u) \otimes \delta l := \chi\hat{\delta}u \otimes \delta l, \\ \chi\delta u \otimes \delta l(\bar{I}) &= \chi\delta u \otimes \delta l(N_1 + \chi u \otimes \hat{l}) = \chi\delta u \otimes (\delta l N_1 + \chi\delta l \cdot u \otimes \hat{l}) := \chi\delta u \otimes \hat{\delta}l, \end{aligned}$$

We use Sublemma C.4 to conclude that $\hat{\delta}u, \hat{\delta}l = O(\mu)$.

Next we consider $(\bar{V})v' \otimes v(\bar{I})$.

(C.6)

$$\begin{aligned} (\bar{V})v' \otimes v(\bar{I}) &= (\bar{V})(-\chi\delta u \otimes \delta l + S)(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)(-\chi\delta u \otimes \delta l + S)(\bar{I}) \\ &= (-\chi\hat{\delta}u \otimes \delta l + (\bar{V})S)(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)(-\chi\delta u \otimes \delta l + S(\bar{I})) \\ &= \chi^2\hat{\delta}u \otimes \delta l[(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)]\delta u \otimes \hat{\delta}l - \chi\hat{\delta}u \otimes \delta l[(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)]S(\bar{I}) \\ &\quad - \chi(\bar{V})S[(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)]\delta u \otimes \hat{\delta}l + (\bar{V})S[(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)]S(\bar{I}) \\ &\lesssim \hat{\delta}u \otimes \hat{\delta}l + \hat{\delta}u \otimes \hat{l}_{i'}(III)S(\bar{I}) + (\bar{V})S(III)\hat{u}_{iii} \otimes \hat{\delta}l + (\bar{V})S(III)\hat{u}_{iii} \otimes \hat{l}_{i'}(III)S(\bar{I}) \end{aligned}$$

where in the last step we use Corollary C.1. The first term above is $O(\mu^2)$. To study the remaining three terms, we continue the calculation in Sublemma C.2 to get

Sublemma C.5. *We have the following estimates.*

$$\begin{aligned} (1) \quad \hat{l}_{i'}(III)S &\lesssim \left(1, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi^2}; \frac{\mu\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \mu, \mu; \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi}\right) = O(1), \\ \hat{l}_{i'}(III)SN_1 &\lesssim \left(1, \mu, \mu, \mu; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \mu, \mu; \mu, \mu\right) = O(1), \\ (2) \quad S(III)\hat{u}_{iii} &\lesssim \left(\frac{\mu\mathcal{G}}{\chi^2}, \mu\mathcal{G}, \frac{\mu\mathcal{G}}{\chi^2}, \frac{\mu\mathcal{G}}{\chi^2}; \mu^2\mathcal{G}, \mu, \frac{\mu\mathcal{G}}{\chi}, \frac{1}{\chi}; 0, \frac{\mu\mathcal{G}}{\chi}\right) = O(\mu\mathcal{G}), \\ N_5S(III)\hat{u}_{iii} &\lesssim \left(\mu^2\mathcal{G}, \mu\mathcal{G}, \mu^2\mathcal{G}, \mu^2\mathcal{G}; \mu^2\mathcal{G}, \mu, \frac{\mu\mathcal{G}}{\chi}, \frac{1}{\chi}; \mu^2\mathcal{G}, \mu^2\mathcal{G}\right) = O(\mu\mathcal{G}). \end{aligned}$$

Corollary C.2. $\hat{l}_{i'}(III)S \cdot u \lesssim \frac{\mu\mathcal{G}}{\chi^2}, \quad l' \cdot S(III)\hat{u}_{iii} \lesssim \frac{\mu^2\mathcal{G}}{\chi}.$

Using the Sublemma C.5 and Corollary C.2, we get

$$\begin{aligned} \hat{l}_{i'}(III)S(\bar{I}) &= \hat{l}_{i'}(III)SN_1 + \chi\hat{l}_{i'}(III)S \cdot u \otimes \hat{l} = O(1), \\ (\bar{V})S(III)\hat{u}_{iii} &= N_5S(III)\hat{u}_{iii} + \chi\hat{u} \otimes l' \cdot S(III)\hat{u}_{iii} = O(\mu\mathcal{G}). \end{aligned}$$

Accordingly the fourth term in (C.6) is $O(\mu\mathcal{G})$ and the other terms are even smaller. Hence $(\bar{V})v' \otimes v(\bar{I}) = O(\mu\mathcal{G}) \ll O(\mu\chi)$.

STEP 4: *The last summand in (C.3) gives the $O(\chi)$ contribution in Lemma 3.2 and a $O(\mu\chi)$ perturbation.*

To proceed, the following calculation is needed.

Sublemma C.6. *We have the following estimates.*

$$\begin{aligned} (1) \quad (III)\delta u &\lesssim \left(\frac{\mu\mathcal{G}}{\chi^3}, \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi^3}, \frac{\mu\mathcal{G}}{\chi^3}; \frac{\mu^2\mathcal{G}}{\chi}, \mu, \frac{\mathcal{G}}{\chi^3}, \frac{1}{\chi}; \frac{1}{\chi}, \frac{1}{\chi}\right)^T = O(\mu), \\ \delta l(III) &\lesssim \left(\frac{\mathcal{G}}{\chi}, \frac{\mathcal{G}}{\chi^3}, \frac{\mathcal{G}}{\chi^3}, \frac{\mathcal{G}}{\chi^3}; \frac{1}{\chi^2}, \frac{1}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \mu; \frac{1}{\chi}, \frac{1}{\chi}\right) = O(\mu), \end{aligned}$$

$$\begin{aligned}
(2) \quad \delta l(III)S &\lesssim \left(\frac{\mathcal{G}}{\chi}, \frac{\mathcal{G}}{\chi^3}, \frac{\mathcal{G}}{\chi^3}, \frac{\mathcal{G}}{\chi^3}; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \mu; \frac{\mu}{\chi}, \frac{\mu}{\chi} \right) = O(\mu), \\
\delta l(III)SN_1 &\lesssim \left(\frac{\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi}; \frac{\mathcal{G}}{\chi^2}, \frac{1}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \mu; \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi} \right) = O(\mu), \\
(3) \quad S(III)\delta u &\lesssim \left(\frac{\mu\mathcal{G}}{\chi^3}, \frac{\mu\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi^3}, \frac{\mu\mathcal{G}}{\chi^3}; \frac{\mu^2\mathcal{G}}{\chi}, \mu, \frac{\mu\mathcal{G}}{\chi^2}, \frac{1}{\chi}; 0, \frac{\mu}{\chi} \right) = O(\mu), \\
N_5S(III)\delta u &\lesssim \left(\frac{\mu^2\mathcal{G}}{\chi}, \frac{\mu\mathcal{G}}{\chi}, \frac{\mu^2\mathcal{G}}{\chi}, \frac{\mu^2\mathcal{G}}{\chi}; \frac{\mu^2\mathcal{G}}{\chi}, \mu, \frac{\mathcal{G}}{\chi^3}, \frac{1}{\chi}; \frac{\mu\mathcal{G}}{\chi^2}, \frac{\mu\mathcal{G}}{\chi} \right) = O(\mu).
\end{aligned}$$

Corollary C.3. (1) $\delta l(III)\delta u \lesssim \frac{\mu}{\chi}$,
(2) $\delta l(III)S \cdot u \lesssim \frac{\mu\mathcal{G}}{\chi^2}$, $l' \cdot S(III)\delta u \lesssim \frac{\mu\mathcal{G}}{\chi^2}$.

We are now ready to consider the last summand in (C.3). Using (C.5), we get

$$\begin{aligned}
&(\bar{V})(-\chi\delta u \otimes \delta l + S)(III)(-\chi\delta u \otimes \delta l + S)(\bar{I}) \\
&= (-\chi\hat{\delta}u \otimes \delta l + (\bar{V})S)(III)(-\chi\delta u \otimes \hat{\delta}l + S(\bar{I})) \\
(C.7) \quad &= \chi^2\hat{\delta}u \otimes \delta l(III)\delta u \otimes \hat{\delta}l - \chi\hat{\delta}u \otimes \delta l(III)S(\bar{I}) - \chi(\bar{V})S(III)\delta u \otimes \hat{\delta}l \\
&+ (\bar{V})S(III)S(\bar{I}).
\end{aligned}$$

The first term in the RHS of (C.7) is $O(\mu^3\chi)$ using Corollary C.3(1). Next,

$$\delta l(III)S(\bar{I}) = \delta l(III)SN_1 + \chi\delta l(III)Su \otimes \hat{l} = O(\mu).$$

This implies the second term in the RHS of (C.7) is $O(\mu\chi)$. To consider the third term in the RHS of (C.7), we note that

$$(\bar{V})S(III)\delta u = N_5S(III)\delta u + \chi\hat{u} \otimes l' \cdot S(III)\delta u = O(\mu).$$

So the third term is also $O(\mu\chi)$. Thus we get

$$(C.7) = (\bar{V})S(III)S(\bar{I}) + O(\mu\chi).$$

We need the following calculations.

Sublemma C.7. *We have the following estimates as $1/\chi \ll \mu \rightarrow 0$.*

$$\begin{aligned}
(1) \quad l'SN_1, l'MSN_1, l'SMSN_1 &= (1, 0_{1 \times 9}) + O(\mu) \rightarrow \hat{\mathbf{1}}_j, \\
(2) \quad N_5Su, N_5MSu, N_5SMSu &= (0, 1, 0_{1 \times 8})^T + O(\mu) \rightarrow \tilde{w}, \\
(3) \quad l'Su, l'S\tilde{u}, l'SMSu &\lesssim \frac{\mu}{\chi}, \quad \tilde{l}Su = O\left(\frac{1}{\chi^2}\right). \\
(4) \quad N_5SMSN_1 &= O(\mu\chi).
\end{aligned}$$

Proof. Again we use a computer. Items (1) and (2) can be obtained by taking limit $\lim_{\mu \rightarrow 0, \chi \rightarrow \infty}$ using Mathematica. In item (4), we use Mathematica to get $\lim_{\mu \rightarrow 0, \chi \rightarrow \infty} N_5SMSN_1/\chi = 0$. \square

To understand (1) and (2) heuristically, we notice that all the entries of l' are small except the first one, so multiplying l' to a matrix corresponds to picking out the first row. Though M, N_1 have some large entries of order $O(\mu\chi)$, the corresponding entries of l' are small enough to suppress them. The first rows of the matrices

S, M, N_1 all have a similar structure to l' . Therefore, we may think l' as a left eigenvector of the matrices. The same heuristic argument applies to u .

To see where (4) comes from we may think of S as identity. The big entries of $O(\mu\chi)$ in M, N_1, N_5 are off-diagonal. It is not hard to keep track of these $O(\mu\chi)$ entries to see that we do not get terms greater than $O(\mu\chi)$.

Next, we multiply $(\bar{V})S(III)S(\bar{I})$ to get

$$\begin{aligned}
& [N_5 + \chi\hat{u} \otimes l']S[M + \chi\tilde{u} \otimes l' + \chi u \otimes \tilde{l}]S[N_1 + \chi u \otimes \hat{l}] \\
&= [N_5 + \chi\hat{u} \otimes l'] [SM + \chi S\tilde{u} \otimes l' + \chi Su \otimes \tilde{l}] [SN_1 + \chi Su \otimes \hat{l}] \\
&\lesssim [N_5 SM + N_5(\chi S\tilde{u} \otimes l' + \chi Su \otimes \tilde{l}) + \chi\hat{u} \otimes l' SM + \mu\chi\hat{u} \otimes l' + O(\mu)] \cdot \\
&\quad (SN_1 + \chi Su \otimes \hat{l}) \\
&= N_5 S M S N_1 + \chi N_5 S \tilde{u} \otimes l' S N_1 + \chi N_5 S u \otimes \tilde{l} S N_1 + \chi\hat{u} \otimes l' S M S N_1 + \\
\text{(C.8)} \quad & \mu\chi\hat{u} \otimes l' S N_1 + \chi N_5 S M S u \otimes \hat{l} + N_5(\chi S\tilde{u} \otimes l' + \chi Su \otimes \tilde{l})(\chi Su \otimes \hat{l}) + \\
& (\chi\hat{u} \otimes l') S M (\chi Su \otimes \hat{l}) + \mu\chi^2\hat{u} \otimes l' \cdot Su \otimes \hat{l} + O(\mu\chi),
\end{aligned}$$

where in \lesssim , we use that $l'Su \sim l'S\tilde{u} \lesssim \mu/\chi$ by Sublemma C.7(3).

The first term in (C.8) is $O(\mu\chi)$ by Sublemma C.7(4).

The ninth term $\mu\chi^2\hat{u} \otimes l' \cdot Su \otimes \hat{l} = O(\mu^2\chi)$ since $l' \cdot Su = O\left(\frac{\mu}{\chi}\right)$ by Sublemma C.7(3).

The seventh term

$$N_5(\chi S\tilde{u} \otimes l' + \chi Su \otimes \tilde{l})(\chi Su \otimes \hat{l}) = O(\mu\chi)$$

using that $\tilde{u} = u + O\left(\frac{1}{\chi}\right)$ and Sublemma C.7(3).

The fifth term has the estimate $\mu\chi\hat{u} \otimes l' S N_1 = O(\mu\chi)$ by Sublemma C.7(1).

The eighth term $(\chi\hat{u} \otimes l') S M (\chi Su \otimes \hat{l}) = O(\mu\chi)$, since $l' S M S u \lesssim \frac{\mu}{\chi}$ by Sublemma C.7(3).

We are left with four terms, the second, third fourth and sixth terms, written together as

$$\text{(C.9)} \quad \chi \left[N_5 S \tilde{u} \otimes l' S N_1 + N_5 S u \otimes \tilde{l} S N_1 + \hat{u} \otimes l' S M S N_1 + N_5 S M S u \otimes \hat{l} \right].$$

We first use the fact that

$$\tilde{u} = u + O\left(\frac{1}{\chi}\right), \quad \tilde{l} = l + O\left(\frac{1}{\chi}\right), \quad l' = l + O\left(\frac{1}{\chi}\right), \quad \hat{l} = l + O(\mu)$$

and $N_1, N_5 = O(\mu\chi)$ to reduce the four terms to

$$\chi [2(N_5 S u + O(\mu)) \otimes (l' S N_1 + O(\mu)) + \hat{u} \otimes l' S M S N_1 + N_5 S M S u \otimes (l' + O(\mu))].$$

Using parts (1) and (2) of Sublemma C.7, we find that each term in expression (C.9) has the form of

$$\chi(u + O(\mu)) \otimes (l' + O(\mu)) = \chi u \otimes l' + O(\mu\chi).$$

Up to now, we have successfully separated the $O(\chi^2)$, $O(\chi)$ and $O(\mu\chi)$ parts in the global map.

Step 5. Completing the proof.

Remember we have dropped the two $O(1)$ matrix $(1 + u_1^i \otimes l_1^i)$ in (I) and the matrix $(1 + u_5^f \otimes l_5^f)$ in (V) in Step 0. We summarize the results of Steps 2 and 4 as follows

$$d\mathbb{G} = (\text{Id}_{10} + u_5^f \otimes l_5^f)(\chi^2 \bar{u}' \otimes \bar{l}' + \chi u \otimes l' + O(\mu\chi))(\text{Id}_{10} + u_1^i \otimes l_1^i).$$

To complete the proof of the lemma, it is enough to define

$$(C.10) \quad \bar{\mathbf{u}} = (\text{Id}_{10} + u_5^f \otimes l_5^f) \bar{u}', \quad \bar{\mathbf{u}} = (\text{Id}_{10} + u_5^f \otimes l_5^f) u, \quad \bar{\mathbf{l}} = \bar{l}' (\text{Id}_{10} + u_1^i \otimes l_1^i), \quad \bar{\mathbf{l}} = l' (\text{Id}_{10} + u_1^i \otimes l_1^i).$$

We obtain the structure of $d\mathbb{G}$ stated in Lemma 3.2. It remains to work out the vectors $\bar{\mathbf{u}}, \bar{\mathbf{u}}, \bar{\mathbf{l}}, \bar{\mathbf{l}}$. We have

$$\bar{\mathbf{u}} = u + O(\mu) \rightarrow (0, 1, 0_{1 \times 8})^T, \quad \bar{\mathbf{l}} = l' + O(\mu) \rightarrow (1, 0_{1 \times 9}) \text{ as } 1/\chi \ll \mu \rightarrow 0$$

using Sublemma C.7 for u, l' and Proposition 5.1 for u_1^i and l_5^f . According to (C.4) in Step 2, we have

$$\bar{u}' = N_5 \hat{u}_{iii'} + O(\mu \mathcal{G}) \hat{u} \quad \bar{l}' = \hat{l}_i N_1.$$

We neglect $O(\mu \mathcal{G}) \hat{u}$ term since it is enough to consider the span $\{N_5 \hat{u}_{iii'}, \hat{u}\}$ and $\hat{u} = u + O(\mu)$ is already provided by the $O(\chi)$ part of $d\mathbb{G}$. Using $\hat{u}_{iii'}$ in Definition C.1 and N_1 in Proposition 5.1, we find in \bar{u}' , we have $N_5 \hat{u}_{iii'} \rightarrow (0, O(1), 0_{1 \times 6}, O(1), O(1))$ as $1/\chi \ll \mu \rightarrow 0$, where the last two $O(1)$ entries are

$$(C.11) \quad (N_5)_{44} \cdot (u_{iii'}(9), u_{iii'}(10))^T = (u_{iii'}(9), u_{iii'}(10))^T = \left(1, -\frac{\hat{L}_4}{\hat{L}_4^2 + \hat{G}_4^2}\right),$$

$((u_{iii'}(9), u_{iii'}(10)))$ is an eigenvector of $(N_5)_{44}$ with eigenvalue 1) and in \bar{l}' , we have

$$\hat{l}_i N_1 \rightarrow \lim l_i = - \left(\frac{\tilde{G}_4}{\tilde{L}_4(\tilde{L}_4^2 + \tilde{G}_4^2)}, 0_{1 \times 7}, -\frac{1}{\tilde{L}_4^2 + \tilde{G}_4^2}, \frac{1}{\tilde{L}_4} \right).$$

It is easy to see that $u \rightarrow \bar{\mathbf{u}}$ using the definition of u in Proposition 5.1. We substitute these calculations back to (C.10) to get $\bar{\mathbf{u}} \rightarrow u_{iii'} + c\bar{\mathbf{u}}$ for some constant c .

Remark C.1. *In this remark, we overview the calculation to explain further the heuristics. As we have remarked in Step 1, the χ^2 part is given by $\hat{u}_{iii'} \otimes \hat{l}_i$ and that χ part is given by $(V)S(III)S(I)$ in (C.3). We neglect the tilde and hat in the following discussion.*

- (1) *For the χ part, in the matrices $(I), (III), (V)$, the $O(\chi)$ entries are at the $(2, 1)$ position which produces the shears exactly in the same way as in [DX], while the block in S corresponding to the x_3 part is $\text{Id}_{4 \times 4}$. So we directly see that the product $(V)S(III)S(I)$ has $O(\chi)$ as the $(2, 1)$ entry. The rest of the entries in $(I), (III), (V)$ are at most $O(\mu\chi)$. These $O(\mu\chi)$ entries are in a good (nilpotent) position, so they do not produce $\mu^2\chi^2$ or higher order terms in the product as we have seen in the analysis of the matrices M, N_1, N_5 in Section 7.2.*

- (2) For the χ^2 part, the two vectors u_{iii} and l_i are the same as the corresponding vectors in [DX] and their $O(1)$ entries involve mainly the x_4 part. To get the χ^2 part, we need $\hat{l}_{i'}(III)\hat{u}_{iii} = O(1)$ and nonzero in (C.2). Notice that the only nonzero entries in u_{iii} are the last two entries. The vector $(III)u_{iii}$ is a linear combination of the last two rows of (III) , which are at most $O(1)$, since in (III) the $O(\chi)$ entry is $(2, 1)$ and the $O(\mu\chi)$ entries have nothing to do with the x_4 part.

To summarize, we get the χ^2 part because the motion of x_4 is separated from x_1, x_3 and we have the χ part since the influence from x_1 is at most $O(\mu\chi)$. Computer is actually not needed to get the two leading χ^2, χ terms in dG , but needed to show the perturbation is at most $O(\mu\chi)$.

Remark C.2. In this remark, we explain the calculation in (C.11), $(u_{iii'}(9), u_{iii'}(10))$ is an eigenvector of $(N_5)_{44}$ with eigenvalue 1. The matrix $(N_5)_{44}$ is the fundamental solution of equation (7.4)

$$\mathbb{V}' = \mathbb{A}\mathbb{V}, \quad \text{where } \mathbb{A} = \begin{bmatrix} \frac{\partial^2 x_4}{\partial G \partial g} \cdot W & \frac{\partial^2 x_4}{\partial g^2} \cdot W \\ -\frac{\partial^2 x_4}{\partial G^2} \cdot W & -\frac{\partial^2 x_4}{\partial G \partial g} \cdot W \end{bmatrix},$$

for some constant vector W up to a time reparametrization. The matrix \mathbb{A} is a nilpotent matrix so that the solution is $(N_5)_{44} = \text{Id} + t\mathbb{A}$. Notice (see Lemma A.3 and the proof of Sublemma 9.2)

$$(u_{iii'}(9), u_{iii'}(10)) = \left(1, -\frac{\hat{L}_4}{\hat{L}_4^2 + \hat{G}_4^2}\right) = \left(1, \text{sign}(u) \frac{\partial g}{\partial G}\right),$$

and $\text{sign}(u) = -1$ for the piece (V) . This gives $\mathbb{A}(u_{iii'}(9), u_{iii'}(10))^T = 0$, so that $(u_{iii'}(9), u_{iii'}(10))$ is an eigenvector of $(N_5)_{44}$ with eigenvalue 1.

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